MATH1050 De Moivre's Theorem and roots of unity

1. **Definition.**
   Let z be a complex number.

   The **modulus** *of z, which we denote by $|z|$, is defined by $|z| = \sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}$.*

   The expression $z = |z|(\cos(\theta) + i\sin(\theta))$ (for some appropriate real number θ) is called the **polar form** *of z.*

   If $z \neq 0$, then such a number θ is called an **argument** *for z. Furthermore, if $-\pi < \theta \leq \pi$, then θ is called the* **principal argument** *of z.*

   Remark. That this definition makes sense is guaranteed by the statement below, which needs be justified carefully:

   - *Let z be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.*

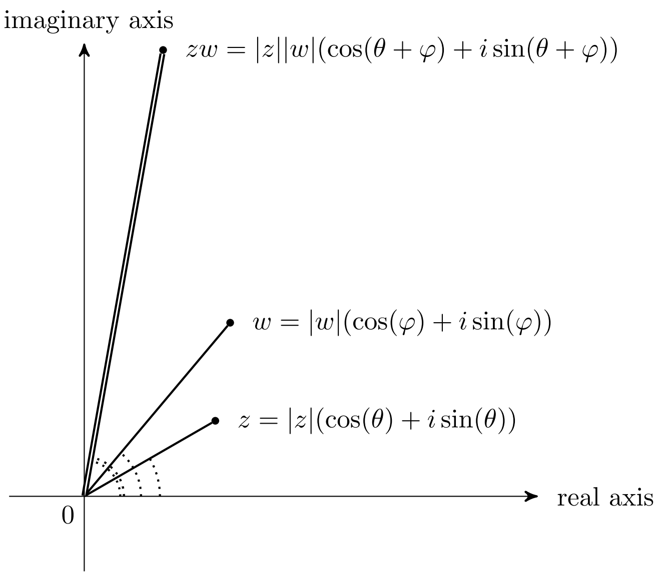
   Further remark. Multiplication and division for complex numbers can be given a nice geometric interpretation in terms of polar form:

   Suppose z, w are non-zero complex numbers, with arguments θ, φ respectively. Then:

   (a) *$zw = |z||w|(\cos(\theta + \varphi) + i\sin(\theta + \varphi))$, and $\dfrac{z}{w} = \dfrac{|z|}{|w|}(\cos(\theta - \varphi) + i\sin(\theta - \varphi))$.*

   (b) *The modulus of zw is $|z||w|$, and the modulus of $\dfrac{z}{w}$ is $\dfrac{|z|}{|w|}$.*

   (c) *$\theta + \varphi$ is an argument for zw, and $\theta - \varphi$ is an argument for $\dfrac{z}{w}$.*



2. **Lemma (1). (Special case of De Moivre's Theorem.)**
   Let θ be a real number. For any $n \in \mathbb{N}\backslash\{0\}$, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

   Proof. Let θ be a real number.

   - For any $n \in \mathbb{N}\backslash\{0\}$, denote by $P(n)$ the proposition $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.
   - $(\cos(\theta) + i\sin(\theta))^1 = \cos(1 \cdot \theta) + i\sin(1 \cdot \theta)$. Then $P(1)$ is true.
   - Let $k \in \mathbb{N}\backslash\{0\}$. Suppose $P(k)$ is true. Then $(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta)$.
     We prove that $P(k+1)$ is true:

$$
\begin{aligned}
(\cos(\theta) + i\sin(\theta))^{k+1} &= (\cos(\theta) + i\sin(\theta))^k(\cos(\theta) + i\sin(\theta)) \\
&= (\cos(k\theta) + i\sin(k\theta))(\cos(\theta) + i\sin(\theta)) \\
&= (\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + i(\sin(k\theta)\cos(\theta) + \cos(k\theta)\sin(\theta)) \\
&= \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos((k+1)\theta) + i\sin((k+1)\theta)
\end{aligned}
$$

     Hence $P(k+1)$ is true.
   - By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0\}$.

3. **De Moivre's Theorem.**
Let θ be a real number. For any $m \in \mathbb{Z}$, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

Proof. Let θ be a real number. Let $m \in \mathbb{Z}$.

- (Case 1). Suppose $m = 0$. Then

$$(\cos(\theta) + i\sin(\theta))^m = (\cos(\theta) + i\sin(\theta))^0 = 1 = (\cos(0 \cdot \theta) + i\sin(0 \cdot \theta)) = \cos(m\theta) + i\sin(m\theta).$$

- (Case 2). Suppose $m > 0$. By Lemma (1), we have $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.
- (Case 3). Suppose $m < 0$. Define $n = -m$. Then $n \in \mathbb{N}\backslash\{0\}$. Therefore

$$(\cos(\theta) + i\sin(\theta))^m = \frac{1}{(\cos(\theta) + i\sin(\theta))^n} = \frac{1}{\cos(n\theta) + i\sin(n\theta)} = \cos(n\theta) - i\sin(n\theta) = \cos(m\theta) + i\sin(m\theta).$$

Hence in any case, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

4. **Definition.**
Let ζ be a complex number. Let n be a positive integer. ζ is called an **n-th root of unity** *if $\zeta^n = 1$.*

Remark. ζ is an n-th root of unity iff ζ is a root of the polynomial $z^n - 1$ in the complex numbers.)

5. **Theorem (2).**

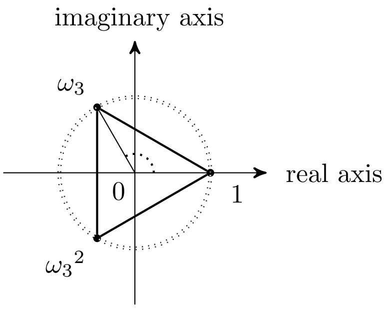
Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) *ω_n is an n-th root of unity.*

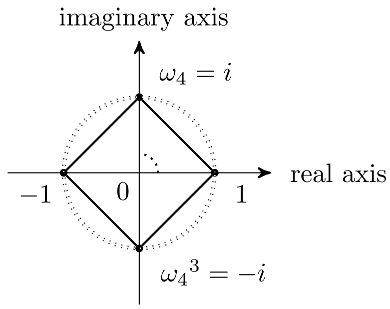
(b) *The n-th roots of unity are the n complex numbers of modulus 1, given by 1, ω_n, ${\omega_n}^2$, ..., ${\omega_n}^{n-1}$.*

Remark. How to visualize these n numbers in terms of plane geometry? They are the n vertices of the regular n-sided polygon inscribed in the unit circle with centre 0 in the Argand plane, with one vertex at the point 1.

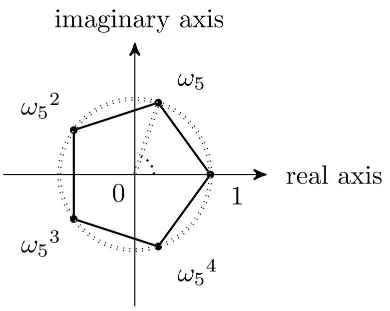
$n = 3$:



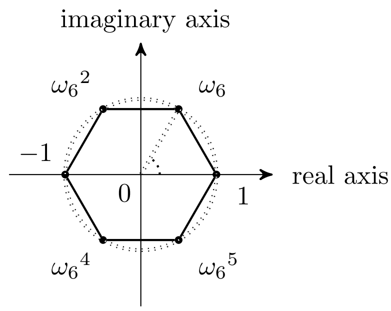
$n = 4$:



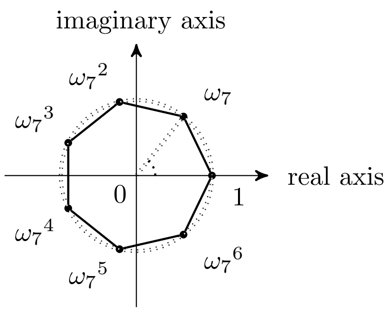
$n = 5$:



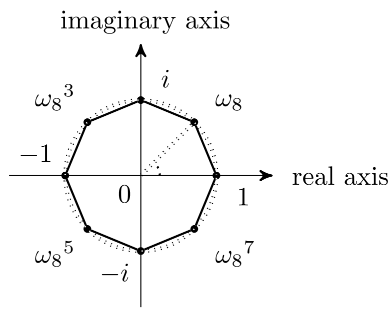
$n = 6$:



$n = 7$:



$n = 8$:



6. **Tacit assumption needed in the argument for Theorem (2).**
A tacit assumption, known as **Division Algorithm for integers**, is used in the argument. It reads:

Let $u, v \in \mathbb{Z}$. Suppose $v \neq 0$. Then there exist some unique $q, r \in \mathbb{Z}$ such that $u = qv + r$ and $0 \leq r < |v|$.

7. **Proof of Theorem (2).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) By De Moivre's Theorem, we have $(\omega_n)^n = (\cos(n\theta_n) + i\sin(n\theta_n)) = \cos(2\pi) + i\sin(2\pi) = 1$.

(b) i. For each $k = 0, 1, 2, \cdots, n-1$, we have $({\omega_n}^k)^n = ({\omega_n}^n)^k = 1^k = 1$.

ii. Let ζ be a complex number. Suppose ζ is an n-th root of unity. Then $\zeta^n = 1$. [We want to deduce that $\zeta = {\omega_n}^r$ for some $r \in [\![0, n-1]\!]$.]
We have $|\zeta|^n = 1$. Then $|\zeta| = 1$. ζ has an argument, say, φ. Therefore $\zeta = \cos(\varphi) + i\sin(\varphi)$.
By De Moivre's Theorem, we have $1 = \zeta^n = (\cos(\varphi) + i\sin(\varphi))^n = (\cos(n\varphi) + i\sin(n\varphi))$.
Then $\cos(n\varphi) = 1$ and $\sin(n\varphi) = 0$. Therefore there exists some $m \in \mathbb{Z}$ such that $n\varphi = 2m\pi$.
Now $\varphi = \dfrac{m}{n} \cdot 2\pi = m\theta_n$.
By Division Algorithm for the integers, there exist some $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r < n$.
Then we have $\varphi = m\theta_n = (qn + r)\theta_n = qn\theta_n + r\theta_n = 2q\pi + r\theta_n$.
Therefore $\zeta = \cos(\varphi) + i\sin(\varphi) = \cos(r\theta_n) + i\sin(r\theta_n) = {\omega_n}^r$.

8. **Corollary (3).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

The polynomial $z^n - 1$ with indeterminate z is completely factorized as $z^n - 1 = (z-1)(z-\omega_n)(z-{\omega_n}^2)\cdot\ldots\cdot(z-{\omega_n}^{n-1})$.

Proof. Exercise. (Apply Factor Theorem.)

Remark. In fact, the polynomial $z^n - 1$ can be factorized as a product of finitely many quadratic polynomials with real coefficients

$$z^2 - 2z\cos(\theta_n) + 1, \quad z^2 - 2z\cos(2\theta_n) + 1, \quad z^2 - 2z\cos(3\theta_n) + 1, \cdots$$

and the linear polynomial $z - 1$ and, when n is even, also together with the linear polynomial $z + 1$. (The argument starts with the observation that ${\omega_n}^{-1} = \overline{\omega_n}$. Why? How?)

9. **Definition.**
*Let n be a positive integer. Let w, ζ be complex numbers. ζ is said to be an **n-th root** of w if $\zeta^n = w$.*
Remark. ζ is an n-th root of w iff ζ is a root of the polynomial $z^n - w$ in the complex numbers.

10. **Lemma (4).**
Let n be a positive integer. Let w be a non-zero complex number. Suppose φ is an argument for w.
Then $\zeta = \sqrt[n]{|w|}\left(\cos\left(\dfrac{\varphi}{n}\right) + i\sin\left(\dfrac{\varphi}{n}\right)\right)$ is an n-th root of w.
Proof. Exercise. (Apply De Moivre's Theorem.)

11. **Theorem (5).**

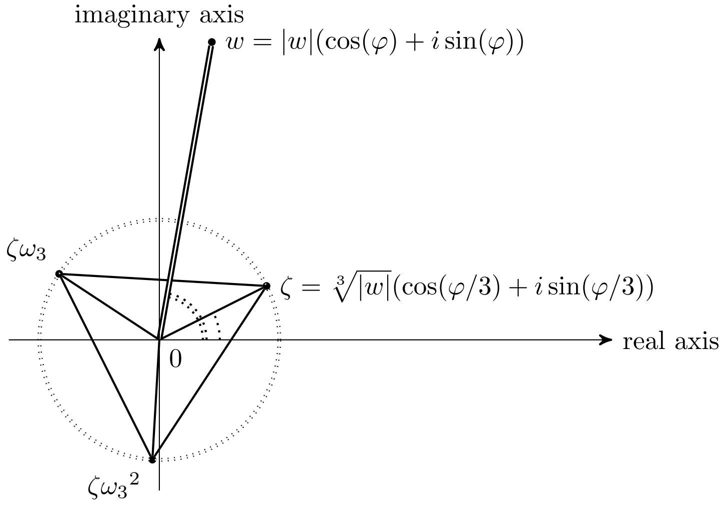
Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

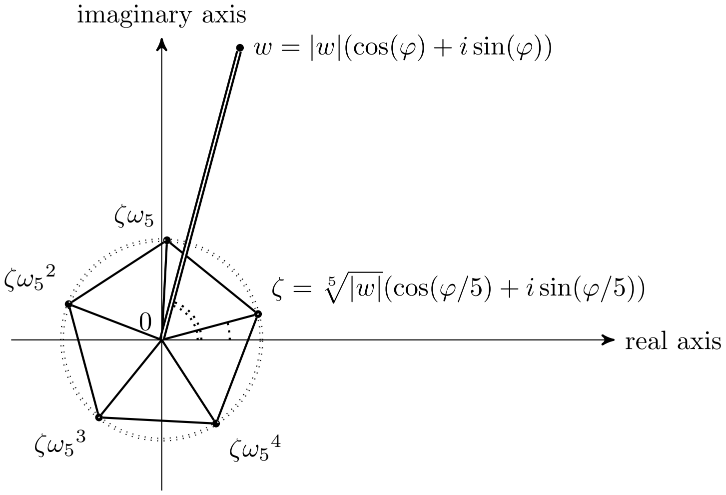
The n-th roots of w are the n complex numbers given by $\zeta, \zeta\omega_n, \zeta{\omega_n}^2, \cdots, \zeta{\omega_n}^{n-1}$.

Remark. How to visualize these n numbers in terms of plane geometry? They are the n vertices of the regular n-sided polygon inscribed in the circle with centre 0 and radius $\sqrt[n]{|w|}$ in the Argand plane, with one vertex at the point ζ.

- Cubic roots:



- Quintic roots:



12. **Proof of Theorem (5).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

- We have $\zeta^n = w$.
  For each $n = 0, 1, 2, \cdots, n-1$, we have $({\omega_n}^k)^n = 1$. Then $(\zeta{\omega_n}^k)^n = \zeta^n({\omega_n}^n)^k = 1 \cdot 1^k = 1$.
- Let ρ be a complex number. Suppose ρ is an n-th root of w.
  Then $\rho^n = w$. We have $\left(\dfrac{\rho}{\zeta}\right)^n = \dfrac{\rho^n}{\zeta^n} = \dfrac{w}{w} = 1$.
  Then $\dfrac{\rho}{\zeta}$ is an n-th root of unity. Therefore there exists some $r = 0, 1, 2, \cdots, n-1$ such that $\dfrac{\rho}{\zeta} = {\omega_n}^r$. For the same r, we have $\rho = \zeta{\omega_n}^r$.

13. **Corollary (6).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

The polynomial $z^n - w$ with indeterminate z is completely factorized as

$$z^n - w = (z - \zeta)(z - \zeta\omega_n)(z - \zeta{\omega_n}^2) \cdot \ldots \cdot (z - \zeta{\omega_n}^{n-1}).$$

Proof. Exercise. (Apply Factor Theorem.)