1. **Definition.**

Let z be a complex number.

The **modulus** *of z, which we denote by $|z|$, is defined by $|z| = \sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}$.*

The expression $z = |z|(\cos(\theta) + i\sin(\theta))$ (for some appropriate real number θ) is called the **polar form** *of z.*

If $z \neq 0$, then such a number θ is called an **argument** *for z. Furthermore, if $-\pi < \theta \leq \pi$, then θ is called the* **principal argument** *of z.*

Remark. This definition makes sense is guaranteed by the statement below, which needs be justified carefully:

- *Let z be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.*

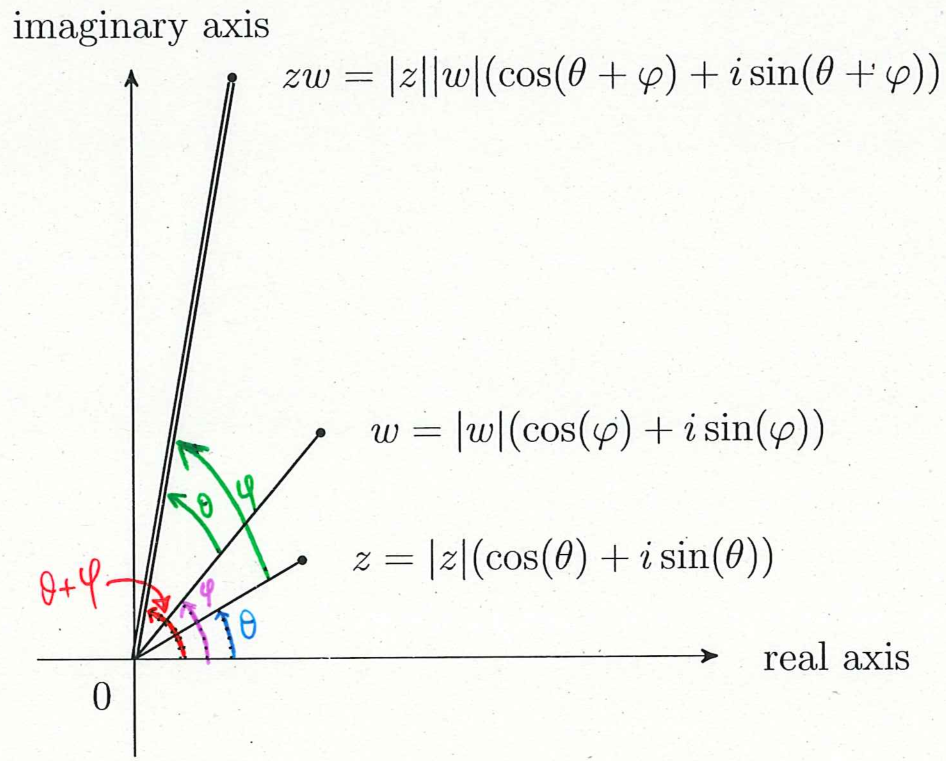
Further remark. Multiplication and division for complex numbers can be given a nice geometric interpretation in terms of polar form:

Suppose z, w are non-zero complex numbers, with arguments θ, φ respectively. Then:

(a) $zw = |z||w|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$, *and ...*

(b) *The modulus of zw is $|z||w|$, and ...*

(c) $\theta+\varphi$ *is an argument for zw, and ...*



2. **Lemma (1). (Special case of De Moivre's Theorem.)**

Let θ be a real number. For any $n \in \mathbb{N}\backslash\{0\}$, $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$.

Proof. Let θ be a real number.

- For any $n \in \mathbb{N}\backslash\{0\}$, denote by $P(n)$ the proposition
$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta).$$
- $(\cos(\theta) + i\sin(\theta))^1 = \cos(1 \cdot \theta) + i\sin(1 \cdot \theta)$. Then $P(1)$ is true.
- Let $k \in \mathbb{N}\backslash\{0\}$. Suppose $P(k)$ is true. Then $(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta)$.

  We prove that $P(k+1)$ is true:
$$\begin{aligned}
&(\cos(\theta) + i\sin(\theta))^{k+1} \\
= \ &(\cos(\theta) + i\sin(\theta))^k(\cos(\theta) + i\sin(\theta)) \\
= \ &(\cos(k\theta) + i\sin(k\theta))(\cos(\theta) + i\sin(\theta)) \\
= \ &(\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)) + i(\sin(k\theta)\cos(\theta) + \cos(k\theta)\sin(\theta)) \\
= \ &\cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos((k+1)\theta) + i\sin((k+1)\theta)
\end{aligned}$$
  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0\}$.

3. **De Moivre's Theorem.**

Let θ be a real number. For any $m \in \mathbb{Z}$, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

Proof. Let θ be a real number. Let $m \in \mathbb{Z}$.

- (Case 1). Suppose $m = 0$. Then

  $$(\cos(\theta)+i\sin(\theta))^m = (\cos(\theta)+i\sin(\theta))^0 = 1 = (\cos(0\cdot\theta)+i\sin(0\cdot\theta)) = \cos(m\theta)+i\sin(m\theta).$$

- (Case 2). Suppose $m > 0$.

  By Lemma (1), we have $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

- (Case 3). Suppose $m < 0$. Define $n = -m$. Then $n \in \mathbb{N}\backslash\{0\}$. Therefore

$$\begin{aligned}
(\cos(\theta) + i\sin(\theta))^m &= \frac{1}{(\cos(\theta) + i\sin(\theta))^n} \\
\text{By Lemma(1)} \rightarrow &= \frac{1}{\cos(n\theta) + i\sin(n\theta)} \\
\text{By `}z\bar{z}=|z|^2\text{'} \rightarrow &= \cos(n\theta) - i\sin(n\theta) \\
&= \cos(m\theta) + i\sin(m\theta).
\end{aligned}$$

Hence in any case, $(\cos(\theta) + i\sin(\theta))^m = \cos(m\theta) + i\sin(m\theta)$.

4. **Definition.**

Let ζ be a complex number. Let n be a positive integer. ζ is called an **n-th root of unity** *if $\zeta^n = 1$.*

Remark. ζ is an n-th root of unity iff ζ is a root of the polynomial $z^n - 1$ in the complex numbers.)

5. **Theorem (2).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) *ω_n is an n-th root of unity.*

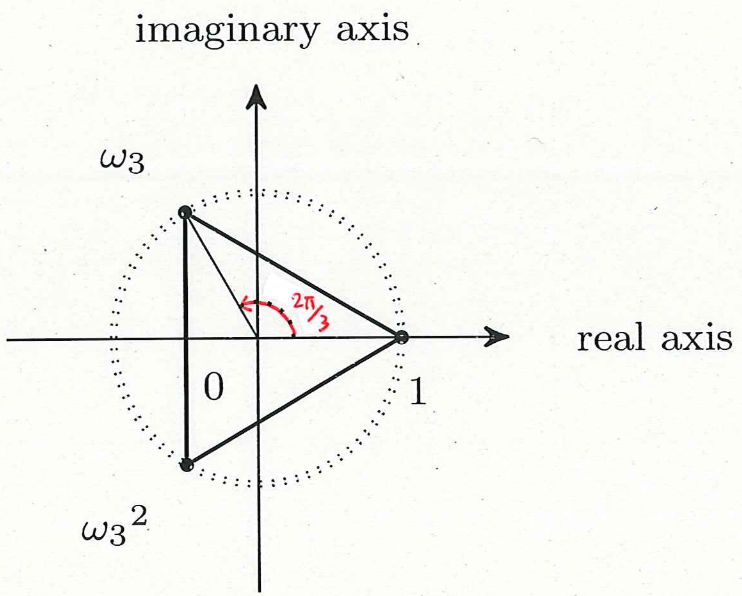
(b) *The n-th roots of unity are the n complex numbers of modulus 1, given by*

$$1, \omega_n, {\omega_n}^2, \ldots, {\omega_n}^{n-1}.$$

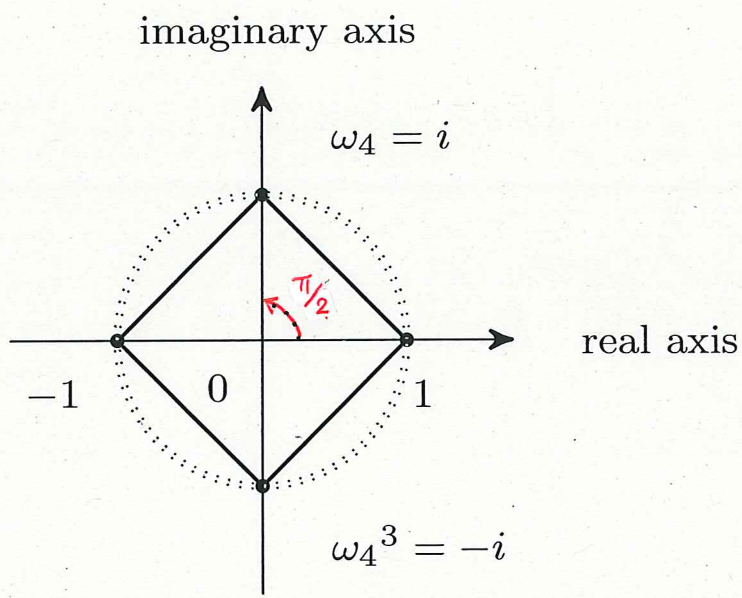
Remark to Theorem (2). How to visualize these n numbers in terms of plane geometry?

They are the n vertices of the regular n-sided polygon inscribed in the unit circle with centre 0 in the Argand plane, with one vertex at the point 1.

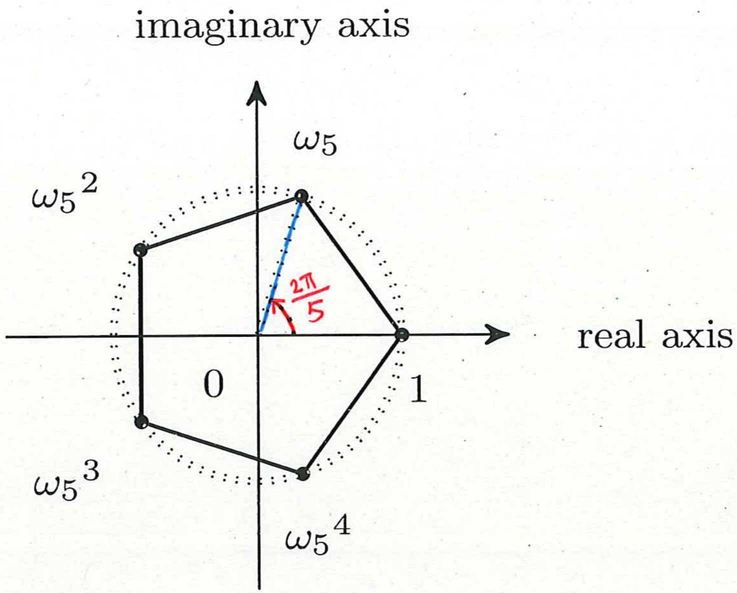
$n = 3$:



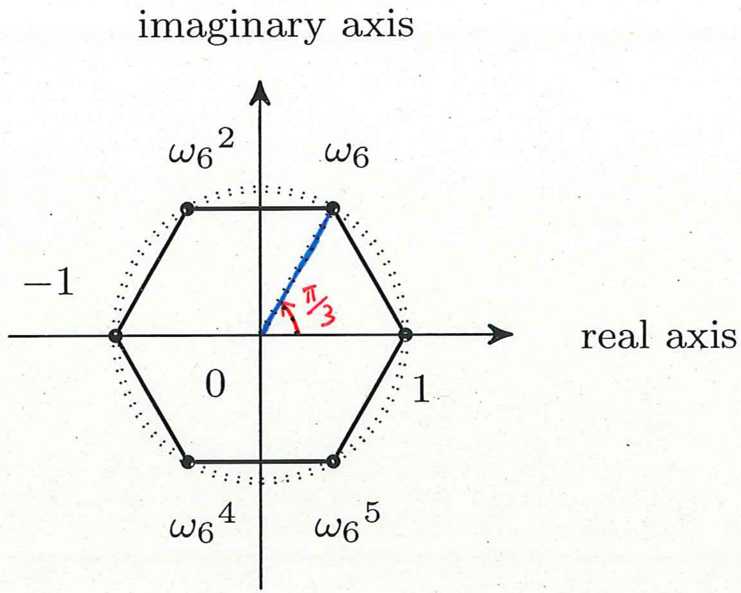
$n = 4$:



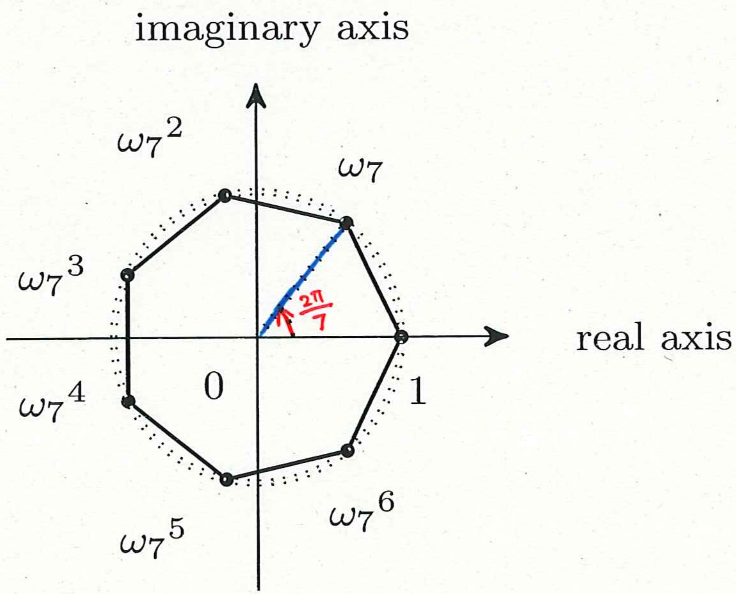
$n = 5$:


imaginary axis
ω5
ω5 2
ω5 3
ω5 4
2π/5
real axis
0
1

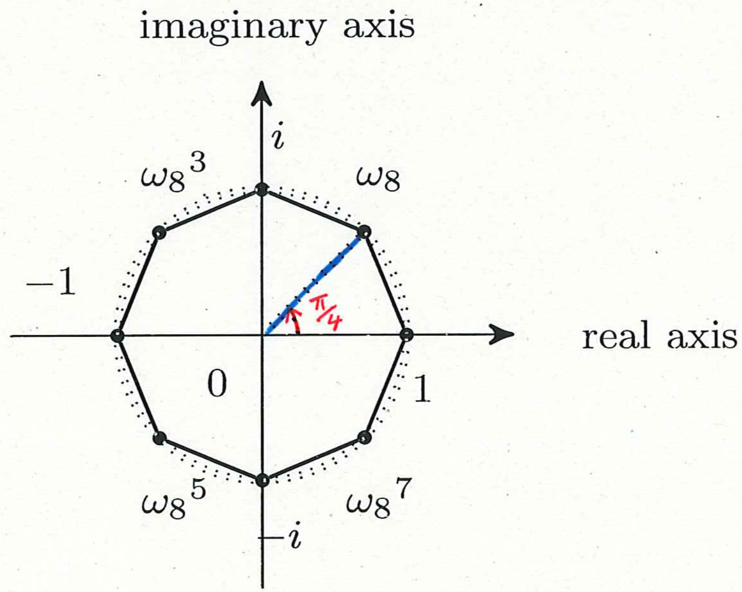
$n = 6$:


imaginary axis
ω6 2
ω6
−1
π/3
real axis
0
1
ω6 4
ω6 5

$n = 7$:


imaginary axis
ω7 2
ω7
ω7 3
2π/7
real axis
ω7 4
0
1
ω7 6
ω7 5

$n = 8$:


imaginary axis
i
ω8 3
ω8
−1
π/4
real axis
0
1
ω8 5
ω8 7
−i

6. **Tacit assumption need in the argument for Theorem (2).**

A tacit assumption, known as **Division Algorithm for integers**, is used in the argument. It reads:

Let $u, v \in \mathbb{Z}$. Suppose $v \neq 0$. Then there exist some unique $q, r \in \mathbb{Z}$ such that $u = qv + r$ and $0 \leq r < |v|$.

7. Proof of Theorem (2).

{Tacitly assumed result to be applied at (✱):
Let $u, v \in \mathbb{Z}$. Suppose $v \neq 0$. Then there exist some unique $q, r \in \mathbb{Z}$ such that $u = qv + r$ and $0 \leq |r| < |v|$.}

Let n be a positive integer. Write $\theta_n = \frac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

(a) By De Moivre's Theorem, $(\omega_n)^n = (\cos(n\theta_n) + i\sin(n\theta_n)) = \cos(2\pi) + i\sin(2\pi) = 1$.

(b) i. For each $k = 0, 1, 2, \cdots, n-1$, we have $(\omega_n{}^k)^n = (\omega_n{}^n)^k = 1^k = 1$.

ii. Let ζ be a complex number. Suppose ζ is an n-th root of unity. Then $\zeta^n = 1$.

[We want to deduce that $\zeta = \omega_n{}^r$ for some $r \in [\![0, n-1]\!]$.]

We have $|\zeta|^n = |\zeta^n| = 1$. Then $|\zeta| = 1$.

ζ has an argument, say, φ. Therefore $\zeta = \cos(\varphi) + i\sin(\varphi)$.

[Ask: what more can we say about φ?]

By De Moivre's Theorem, $1 = \zeta^n = (\cos(\varphi) + i\sin(\varphi))^n = \cos(n\varphi) + i\sin(n\varphi)$.

Then $\cos(n\varphi) = 1$ and $\sin(n\varphi) = 0$.

Therefore there exists some $m \in \mathbb{Z}$ such that $n\varphi = 2m\pi$.

Now $\varphi = \frac{m}{n} \cdot 2\pi = m\theta_n$.

(✱) ⟶ By Division Algorithm, there exist some $q, r \in \mathbb{Z}$ such that $m = qn + r$ and $0 \leq r < n$.

Then $\varphi = m\theta_n = (qn + r)\theta_n = qn\theta_n + r\theta_n = 2q\pi + r\theta_n$

Therefore $\zeta = \cos(\varphi) + i\sin(\varphi) = \cos(r\theta_n) + i\sin(r\theta_n) = \omega_n{}^r$. □

8. **Corollary (3).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$. The polynomial $z^n - 1$ with indeterminate z is completely factorized as

$$z^n - 1 = (z-1)(z-\omega_n)(z-{\omega_n}^2)\cdot \ldots \cdot (z-{\omega_n}^{n-1}).$$

Proof. Exercise. (Apply Factor Theorem.)

Remark. In fact, the polynomial $z^n - 1$ can be factorized as a product of finitely many quadratic polynomials with real coefficients

$$z^2 - 2z\cos(\theta_n) + 1, \quad z^2 - 2z\cos(2\theta_n) + 1, \quad z^2 - 2z\cos(3\theta_n) + 1, \cdots$$

and the linear polynomial $z-1$ and, when n is even, also together with the linear polynomial $z+1$. (The argument starts with the observation that ${\omega_n}^{-1} = \overline{\omega_n}$. Why? How?)

9. **Definition.**

Let n be a positive integer. Let w, ζ be complex numbers. ζ is said to be an **n-th root** *of w if $\zeta^n = w$.*

Remark. ζ is an n-th root of w iff ζ is a root of the polynomial $z^n - w$ in the complex numbers.

10. **Lemma (4).**

Let n be a positive integer. Let w be a non-zero complex number. Suppose φ is an argument for w.

Then $\zeta = \sqrt[n]{|w|}(\cos(\varphi/n) + i\sin(\varphi/n))$ is an n-th root of w.

Proof. Exercise. (Apply De Moivre's Theorem.)

11. **Theorem (5).**

Let n be a positive integer. Write $\theta_n = 2\pi/n$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

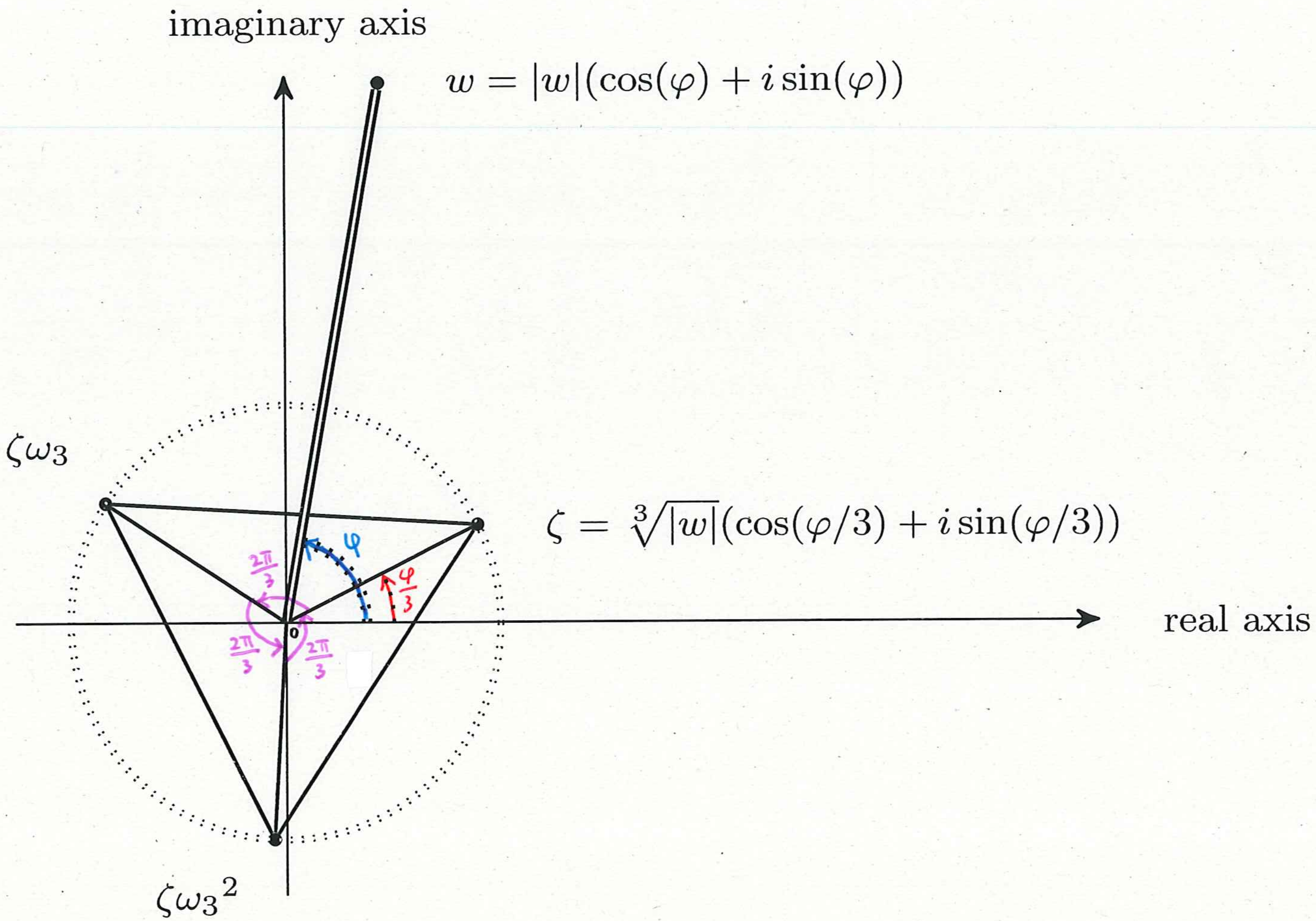
Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

The n-th roots of w are the n complex numbers given by $\zeta, \zeta\omega_n, \zeta{\omega_n}^2, \cdots, \zeta{\omega_n}^{n-1}$.

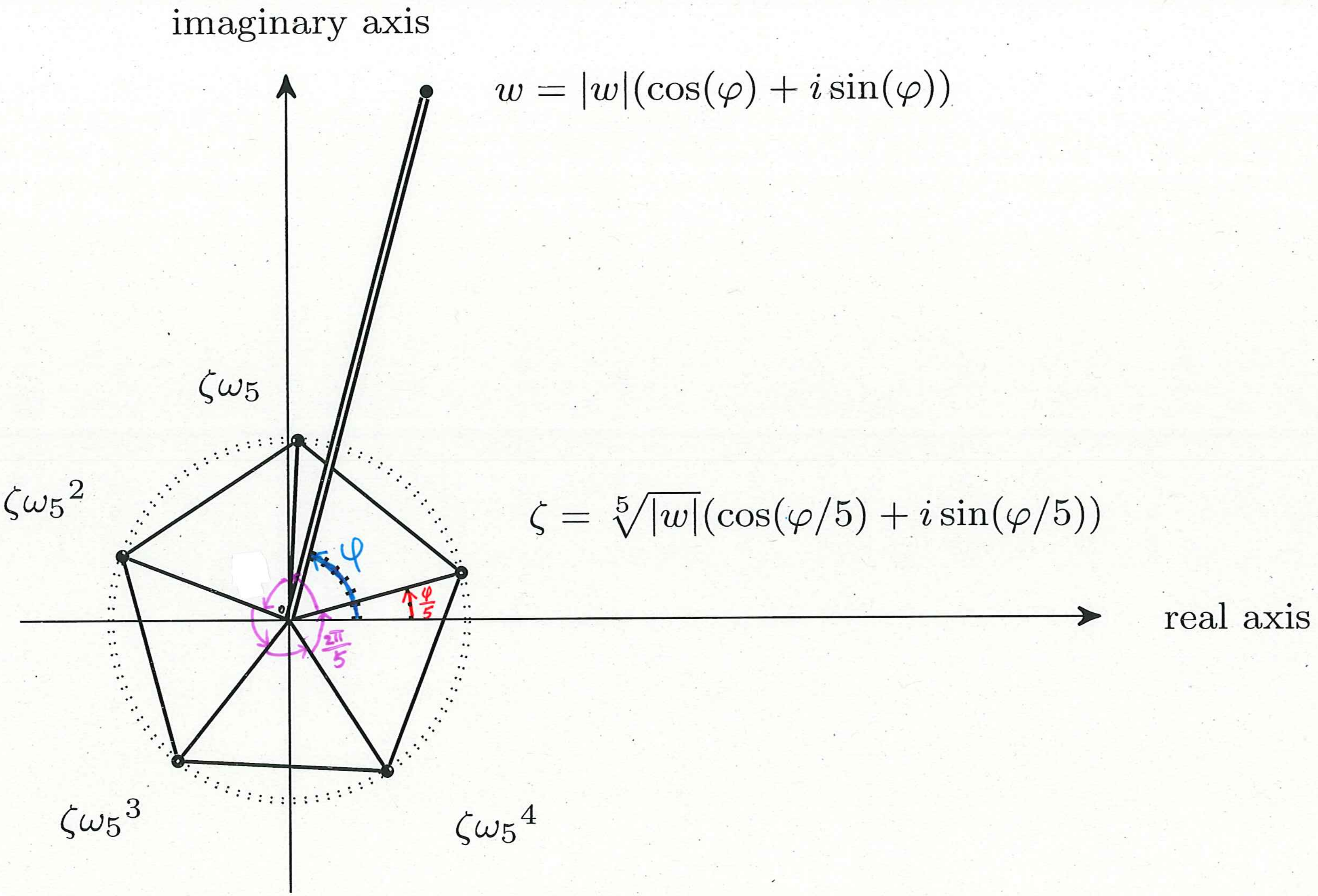
Remark. How to visualize these n numbers in terms of plane geometry?

They are the n vertices of the regular n-sided polygon inscribed in the circle with centre 0 and radius $\sqrt[n]{|w|}$ in the Argand plane, with one vertex at the point ζ.

- Cubic roots:



- Quintic roots:



12. **Proof of Theorem (5).**

Let n be a positive integer. Write $\theta_n = \frac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

- We have $\zeta^n = w$.

  For each $n = 0, 1, 2, \cdots, n-1$, we have $({\omega_n}^k)^n = 1$.

  Then $(\zeta{\omega_n}^k)^n = \zeta^n({\omega_n}^n)^k = 1 \cdot 1^k = 1$.

- Let ρ be a complex number. Suppose ρ is an n-th root of w.

  Then $\rho^n = w$. We have $\left(\frac{\rho}{\zeta}\right)^n = \frac{\rho^n}{\zeta^n} = \frac{w}{w} = 1$.

  Then $\frac{\rho}{\zeta}$ is an n-th root of unity.

  Therefore there exists some $r = 0, 1, 2, \cdots, n-1$ such that $\frac{\rho}{\zeta} = {\omega_n}^r$.

  For the same r, we have $\rho = \zeta{\omega_n}^r$.

13. **Corollary (6).**

Let n be a positive integer. Write $\theta_n = \dfrac{2\pi}{n}$. Define $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$.

Let w be a non-zero complex number, and ζ be an n-th root of w in the complex numbers.

The polynomial $z^n - w$ with indeterminate z is completely factorized as

$$z^n - w = (z - \zeta)(z - \zeta\omega_n)(z - \zeta{\omega_n}^2) \cdot \ldots \cdot (z - \zeta{\omega_n}^{n-1}).$$

Proof. Exercise. (Apply Factor Theorem.)