0. Working knowledge in set language (Method of Specification, subset relations and set equalities) is assumed when it comes to theoretical discussions concerned with the Principle of Mathematical Induction and the Well-ordering Principle for Integers.

1. **What is the Principle of Mathematical Induction?**

   Suppose $P(n)$ is a predicate with variable n. We may form the statement

   ($\star$) '*for any* $n \in \mathbb{N}$, $P(n)$ *holds.*'

   One standard method which may help proving such a statement is 'mathematical induction'. It is based on the validity of **(First) Principle of Mathematical Induction**, in its 'usual' formulation:

   Let $P(n)$ *be a predicate with variable* n.
   Suppose the statement $P(0)$ *is true.*
   Further suppose that for any $k \in \mathbb{N}$, *if the statement* $P(k)$ *is true then the statement* $P(k+1)$ *is true.*
   Then the statement $P(n)$ *is true for any* $n \in \mathbb{N}$.

2. **Format of a mathematical induction argument.**

   The general scheme in the application of the Principle of Mathematical Induction to prove ($\star$) is described below. (For simplicity, we take S to be $\mathbb{N}$ in this description.)

   - Step (0). Identify $P(n)$ and write it down explicitly.
   - Step (1). Prove the statement $P(0)$. (This is the 'initial step argument'.)
   - Step (2). Assume the statement $P(k)$ to be true. (This is called the induction assumption.) Prove the statement $P(k+1)$ under this assumption. (This is the 'induction argument'.)
   - Step (3). Declare that according to the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

   The theoretical support for the above scheme is in the Principle of Mathematical Induction. In Step (1) and Step (2), we are verifying that the predicate $P(n)$ under question satisfies the assumption in the statement of the Principle of Mathematical Induction. In Step (3), we are saying that the conclusion in the statement of the Principle of Mathematical Induction applies to the predicate $P(n)$ under question.

3. **Simple examples in mathematical induction.**

   (a) **Statement (A).**

   $0^3 + 1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ *for any* $n \in \mathbb{N}$.

   Proof of Statement (A).

   - For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition
   $$0^3 + 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$
   - $0^3 = 0 = \dfrac{0^2(0+1)^2}{4}$. Then $P(0)$ is true.
   - Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $0^3 + 1^3 + 2^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$.

     We prove that $P(k+1)$ is true:

     We have
     $$0^3+1^3+2^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2[k^2+4(k+1)]}{4} = \frac{(k+1)^2[(k+1)+1]^2}{4}.$$

     Hence $P(k+1)$ is true.
   - By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

(b) **Statement (B).**

$n^3 - n$ is divisible by 3 for any $n \in \mathbb{N}$.

Proof of Statement (B).

- For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition that
$$n^3 - n \text{ is divisible by } 3.$$
- $0^3 - 0 = 0 = 3 \cdot 0$ and $0 \in \mathbb{Z}$. Hence $0^3 - 0$ is divisible by 3. Then $P(0)$ is true.
- Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $k^3 - k$ is divisible by 3.
  We prove that $P(k+1)$ is true:
  By definition, there exists some $q \in \mathbb{Z}$ such that $k^3 - k = 3q$.
  We have $(k+1)^3 - (k+1) = (k^3 - k) + 3k^2 + 3k = 3(q + k^2 + k)$.
  Note that $q + k^2 + k \in \mathbb{Z}$. Then $(k+1)^3 - (k+1)$ is divisible by 3.
  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

4. **Mathematical Induction not 'starting from 0'?**

There is nothing sacred about the number 0 in mathematical induction. The initial step in an argument by mathematical induction may be concerned with any number other than 0.

(VPMI) Principle of Mathematical Induction, (variant of its 'usual' formulation):

Let $P(n)$ be a predicate with variable n. Let $N \in \mathbb{Z}$.

Suppose the statement $P(N)$ is true.

Further suppose that for any $k \in [\![N, +\infty)$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.

Then the statement $P(n)$ is true for any $n \in [\![N, +\infty)$.

Remark. $[\![N, +\infty)$ is defined to be the set $\{m \in \mathbb{Z} : m \geq N\}$. It is the collection of all integers greater than or equal to N.

Theorem (1).

The statements (UPMI), (VPMI) are logically equivalent to each other.

Proof of Theorem (1). (This is a boring and tedious exercise in word game.)

- Suppose (VPMI) holds. Then (UPMI) holds immediately. (Why?)
- Suppose (UPMI) holds.
  Let $P(n)$ be a predicate with variable n. Let $N \in \mathbb{Z}$.
  Suppose the statement $P(N)$ is true.
  Further suppose that for any $k \in [\![N, +\infty)$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true.
  For each $m \in \mathbb{N}$, denote by $Q(m)$ the predicate $P(N+m)$. By definition, $Q(0)$ is true.
  Let $j \in \mathbb{N}$. Suppose $Q(j)$ is true. Then by definition, $P(n+j)$ is true. Therefore, by assumption, $P(N+j+1)$ is true. Hence by definition, the statement $Q(j+1)$ is true.
  It follows from (UPMI) that $Q(m)$ is true for any $m \in \mathbb{N}$.
  Hence $P(n)$ is true for any $n \in [\![N, +\infty)$.

The result follows.

Further Remark. What we have just seen is a particular instance of a more general situation. Suppose $T(x)$ is a predicate with variable x and W is a set which may be regarded as a 'copy' of $\mathbb{N}$, in the sense that we may label exhaustively and without repetition the elements of S as $x_0, x_1, x_2, \cdots$, we may form the predicate $T(x_n)$ with variable n, which we now denote by $T'(n)$. The statement

($\dagger$) '*for any $x \in W$, $T(x)$ holds*'

is equivalent to

($\dagger'$) '*for any $n \in \mathbb{N}$, $T'(n)$ holds.*'

Hence we may prove ($\dagger$) by applying mathematical induction to prove ($\dagger'$).

5. **Further examples in mathematical induction.**

(a) **Statement (C).**

$\dfrac{(2n)!}{(n!)^2} > \dfrac{4^n}{n+1}$ *for any* $n \in \mathbb{N}\backslash\{0,1\}$.

Proof of Statement (C).

- For any $n \in \mathbb{N}\backslash\{0,1\}$, denote by $P(n)$ the proposition

$$\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}.$$

- Note that $\dfrac{(2\cdot 2)!}{(2!)^2} = \dfrac{24}{4} = 6 > \dfrac{16}{3} = \dfrac{4^2}{2+1}$. Then $P(2)$ is true.

- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true. Then $\dfrac{(2k)!}{(k!)^2} > \dfrac{4^k}{k+1}$. Note that $\dfrac{(2k)!}{(k!)^2} \cdot \dfrac{k+1}{4^k} > 1$.

  We prove that $P(k+1)$ is true:

  [We intend to deduce the inequality $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > \dfrac{4^{k+1}}{(k+1)+1}$.]

  Note that $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > 0$, $\dfrac{4^{k+1}}{(k+1)+1} > 0$ and

$$\begin{aligned}\frac{[2(k+1)]!}{[(k+1)!]^2} \cdot \frac{(k+1)+1}{4^{k+1}} &= \frac{(2k+2)(2k+1)}{(k+1)^2} \cdot \frac{(2k)!}{(k!)^2} \cdot \frac{k+1}{4^k} \cdot \frac{k+2}{4(k+1)} \\ &= \frac{(2k)!}{(k!)^2} \cdot \frac{k+1}{4^k} \cdot \frac{(2k+1)(k+2)}{2(k+1)^2} \\ &> 1 \cdot \frac{(2k+1)(k+2)}{2(k+1)^2} = \frac{2k^2+5k+2}{2k^2+4k+2} > 1\end{aligned}$$

  Then $\dfrac{[2(k+1)]!}{[(k+1)!]^2} > \dfrac{4^{k+1}}{(k+1)+1}$.

  Hence $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

(b) **Statement (D).**

For any $n \in [\![8,+\infty)$, *there exist some* $u, v \in \mathbb{N}$ *such that* $n = 3u + 5v$.

Proof of Statement (D).

- For any $n \in [\![8,+\infty)$, denote by $P(n)$ the proposition that

  there exist some $u, v \in \mathbb{N}$ such that $n = 3u + 5v$.

- Note that $1 \in \mathbb{N}$ and $8 = 3 \cdot 1 + 5 \cdot 1$. Then $P(8)$ is true.

- Let $k \in [\![8,+\infty)$. Suppose $P(k)$ is true. Then there exist some $u, v \in \mathbb{N}$ such that $k = 3u + 5v$.

  We prove that $P(k+1)$ is true:

  Note that $v = 0$ or $v \geq 1$.

  * (Case 1). Suppose $v = 0$. Then $k = 3u$.
    Since $k \geq 8$, we have $u \geq 3$. Then $u - 3 \in \mathbb{N}$.
    Note that $k + 1 = 3u + 1 = 3(u-3) + 5 \cdot 2$.
  * (Case 2). Suppose $v \geq 1$. Then $v - 1 \in \mathbb{N}$.
    Since $u \in \mathbb{N}$, we have $u + 2 \in \mathbb{N}$.
    Note that $k + 1 = 3u + 5v + 1 = 3(u+2) + 5(v-1)$.

  Hence, in any case $P(k+1)$ is true.

- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in [\![8,+\infty)$.

(c) **Statement (E).**

Let $n \in \mathbb{N}\backslash\{0,1\}$. *Suppose* $b_1, b_2, \cdots, b_n \in (0,1)$. *Then* $(1-b_1)(1-b_2)\cdot \ldots \cdot(1-b_n) > 1 - (b_1 + b_2 + \cdots + b_n)$.

Remark. This is one of **Weierstrass's Product Inequalities**.

Proof of Statement (E).

- For any $n \in \mathbb{N}\backslash\{0,1\}$, denote by $P(n)$ the following proposition:

  Suppose $b_1, b_2, \cdots, b_n \in (0,1)$. Then $(1-b_1)(1-b_2)\cdot ... \cdot(1-b_n) > 1-(b_1+b_2+\cdots+b_n)$.
- Suppose $a, b \in (0,1)$. Then $(1-a)(1-b) = 1-a-b+ab > 1-(a+b)$ (because $ab > 0$). It follows that $P(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true. (Therefore, for any $c_1, c_2, \cdots, c_k \in (0,1)$, $(1-c_1)(1-c_2)\cdot ... \cdot(1-c_k) > 1-(c_1+c_2+\cdots+c_k)$.)

  We prove that $P(k+1)$ is true:

  Suppose $b_1, b_2, \cdots, b_k, b_{k+1} \in (0,1)$.

  Then we have

$$\begin{aligned} & (1-b_1)(1-b_2)\cdot ... \cdot(1-b_k)(1-b_{k+1}) \\ > & [1-(b_1+b_2+\cdots+b_k)](1-b_{k+1}) \\ = & 1-(b_1+b_2+\cdots+b_k)-b_{k+1}+(b_1+b_2+\cdots b_k)b_{k+1} \\ > & 1-(b_1+b_2+\cdots+b_k+b_{k+1}). \end{aligned}$$

  (Note that $(b_1+b_2+\cdots b_k)b_{k+1} > 0$.)

  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

(d) **Statement (F)**.

Let p be a prime number. Let $n \in \mathbb{N}\backslash\{0,1\}$. Let $a_1, a_2, \cdots, a_n \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_n$ is divisible by p. Then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p.

Remarks.

- Euclid's Lemma is tacitly assumed here in the argument: *Let $h, k \in \mathbb{Z}$, and p be a prime number. Suppose hk is divisible by p. Then at least one of h, k is divisible by p.*
- The statement to be proved may be thought of as a generalization of Euclid's Lemma from the situation for products of two integers to that for any finite products of integers.

Proof of Statement (F).

- Let p be a prime number. For any $n \in \mathbb{N}\backslash\{0,1\}$, denote by $P(n)$ the following proposition:

  Let $a_1, a_2, \cdots, a_n \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_n$ is divisible by p. Then at least one of $a_1, a_2, \cdots, a_n$ is divisible by p.
- Let $a, b \in \mathbb{Z}$. Suppose ab is divisible by p. Then, by Euclid's Lemma, at least one of a, b is divisible by p. It follows that $P(2)$ is true.
- Let $k \in \mathbb{N}\backslash\{0,1\}$. Suppose $P(k)$ is true. (Therefore, for any $c_1, c_2, \cdots, c_k \in \mathbb{Z}$, if $c_1 c_2 \cdot ... \cdot c_k$ is divisible by p then at least one of $c_1, c_2, \cdots, c_k$ is divisible by p.)

  We prove that $P(k+1)$ is true:

  Let $a_1, a_2, \cdots, a_k, a_{k+1} \in \mathbb{Z}$. Suppose $a_1 a_2 \cdot ... \cdot a_k a_{k+1}$ is divisible by p.

  Note that $a_1 a_2 \cdot ... \cdot a_k \in \mathbb{Z}$ and $(a_1 a_2 \cdot ... \cdot a_k)a_{k+1} = a_1 a_2 \cdot ... \cdot a_k a_{k+1}$.

  Then by Euclid's Lemma, at least one of $a_1 a_2 \cdot ... \cdot a_k$, a_{k+1} is divisible by p.

  * (Case 1). Suppose a_{k+1} is divisible by p. Then at least one of $a_1, a_2, \cdots, a_k, a_{k+1}$, namely a_{k+1}, is divisible by p.
  * (Case 2). Suppose $a_1 a_2 \cdot ... \cdot a_k$ is divisible by p. Then, by $P(k)$, at least one of $a_1, a_2, ..., a_k$ is divisible by p. Therefore at least one of $a_1, a_2, \cdots, a_k, a_{k+1}$ is divisible by p.

  Hence $P(k+1)$ is true.
- By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}\backslash\{0,1\}$.

6. **Why is the Principle of Mathematical Induction true?**

The validity of the Principle of Mathematical Induction rests on something which is intuitively so obvious that we would have never doubted since childhood:

- There is a smallest number in each (non-empty) collection of natural numbers.

We proceed to formulate this 'intuitively obvious' 'fact' in precise mathematical language.

Definition.

Let T be a subset of $\mathbb{R}$, and $\lambda \in T$.

λ is said to be a least element of T if for any $x \in T$, $\lambda \leq x$.

Well-ordering Principle for Integers (WOPI).

Let T be a non-empty subset of $\mathbb{N}$. T has a least element.

Theorem (2).

The Principle of Mathematical Induction (UPMI) is logically equivalent to the Well-ordering Principle for Integers (WOPI).

Proof of Theorem (2). This is a consequence of Theorem (6).

7. **Second Principle of Mathematical Induction.**

The statement below is known as the **Second Principle of Mathematical Induction (PMI2)**:

Let $Q(n)$ be a predicate with variable n.

Suppose the statement $Q(0)$ is true.

Further suppose that for any $k \in \mathbb{N}$, if the statements $Q(0), Q(1), \cdots, Q(k)$ are true then the statement $Q(k+1)$ is true.

Then the statement $Q(n)$ is true for any $n \in \mathbb{N}$.

Theorem (3).

The (First) Principle of Mathematical Induction (UPMI) is logically equivalent to the Second Principle of Mathematical Induction (PMI2).

Proof of Theorem (3). This is a consequence of Theorem (6).

Question. How to prove, say, the statement

($\star$) 'for any $n \in \mathbb{N}$, $Q(n)$ holds.'

with the help of the Second Principle of Mathematical Induction?

Below is the general scheme:

- Step (0). Identify $Q(n)$ and write it down explicitly.
- Step (1). Prove the statement $Q(0)$. (This is the 'initial step argument'.)
- Step (2). Assume the statements $Q(0), Q(1), \cdots, Q(k)$ to be true. (This is called the induction assumption.) Prove the statement $Q(k+1)$ under this assumption. (This is the 'induction argument'.)
- Step (3). Declare that according to the Second Principle of Mathematical Induction, $Q(n)$ is true for any $n \in \mathbb{N}$.

Remark. Give an argument for Statement (F) with the help of the Second Principle of Mathematical Induction.

8. **Principle of Mathematical Induction in set language.**

The Principle of Mathematical Induction (UPMI) can be re-formulated in terms of set equality and subset relations.

(SPMI). Principle of Mathematical Induction, (set-theoretic formulation).

Let S be a subset of $\mathbb{N}$.

Suppose $0 \in S$.

Further suppose that for any $k \in \mathbb{N}$, if $k \in S$ then $k+1 \in S$.

Then $S = \mathbb{N}$.

Theorem (4).

The statements (UPMI), (SPMI) are logically equivalent to each other.

Proof of Theorem (4).

Argument for '(SPMI)$\Longrightarrow$(UPMI)':

Assume (SPMI) holds:

- ∗ *Let S be a subset of $\mathbb{N}$. Suppose $0 \in S$. Further suppose that for any $k \in \mathbb{N}$, if $k \in S$ then $k+1 \in S$. Then $S = \mathbb{N}$.*

[We want to deduce from this assumption '(SPMI) holds' that (UPMI) holds.]

Let $P(n)$ be a predicate with variable n. Suppose the statement $P(0)$ is true. Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true. [We want to deduce, by applying (SPMI), that for any $n \in \mathbb{N}$, the statement $P(n)$ is true.]

Define $S = \{n \in \mathbb{N} : P(n) \text{ is true}\}$. [We now proceed to prove that $S = \mathbb{N}$.]

- Since the statement $P(0)$ is true, we have $0 \in S$.
  Pick any $k \in \mathbb{N}$. Suppose $k \in S$. Then (by the definition of S) the statement $P(k)$ is true. Since $P(k)$ is true, $P(k+1)$ is also true. Therefore (by the definition of S) we have $k+1 \in S$.
  Now, by (SPMI), $S = \mathbb{N}$.

It follows that for any $n \in \mathbb{N}$, the statement $P(n)$ is true.

Argument for '(UPMI)$\Longrightarrow$(SPMI)':

Assume (UPMI) holds:

∗ *Let $P(n)$ be a predicate with variable n. Suppose the statement $P(0)$ is true. Further suppose that for any $k \in \mathbb{N}$, if the statement $P(k)$ is true then the statement $P(k+1)$ is true. Then the statement $P(n)$ is true for any $n \in \mathbb{N}$.*

[We want to deduce from this assumption '(UPMI) holds' that (SPMI) holds.]

Let S be a subset of $\mathbb{N}$. Suppose $0 \in S$. Further suppose that for any $k \in \mathbb{N}$, if $k \in S$ then $k+1 \in S$. [We want to deduce, by applying (UPMI), that $\mathbb{N} \subset S$.]

- For any $n \in \mathbb{N}$, denote by $P(n)$ the proposition $n \in S$. [We now proceed to apply mathematical induction to prove that for any $n \in \mathbb{N}$, $P(n)$ is true.]
  By assumption, $0 \in S$. Then $P(0)$ is true.
  Let $k \in \mathbb{N}$. Suppose $P(k)$ is true. Then $k \in S$. By the assumption on S, since $k \in S$, we also have $k+1 \in S$. Therefore $P(k+1)$ is true.
  By the Principle of Mathematical Induction, $P(n)$ is true for any $n \in \mathbb{N}$.

We have verified that every element of $\mathbb{N}$ is an element of S. Then $\mathbb{N} \subset S$.
By definition $S \subset \mathbb{N}$. Hence $S = \mathbb{N}$.

The result follows.

The Second Principle of Mathematical Induction (UPMI) can also be re-formulated in terms of set equality and subset relations.

(SPMI2). Second Principle of Mathematical Induction, (set-theoretic formulation).

Let U be a subset of $\mathbb{N}$.

Suppose $0 \in U$.

Further suppose that for any $k \in \mathbb{N}$, if $0, 1, 2, \cdots, k \in U$ then $k+1 \in U$.

Then $U = \mathbb{N}$.

Theorem (5).

The statements (PMI2), (SPMI2) are logically equivalent to each other.

Proof of Theorem (5). Exercise. (The argument is similar to that for Theorem (4).)

9. **Theorem (6).**

The Well-ordering Principle for integers (WOPI), the (set-theoretic formulation of the) Principle of Mathematical Induction (SPMI), the (set-theoretic formulation of the) Second Principle of Mathematical Induction (SPMI2) are logically equivalent:

- **Well-ordering Principle for integers (WOPI)**.
  Suppose S be a non-empty subset of $\mathbb{N}$. Then S has a least element.
- **Principle of Mathematical Induction (SPMI)**.
  Let T be a subset of $\mathbb{N}$. Suppose $0 \in T$. Further suppose that for any $k \in \mathbb{N}$, if $k \in T$ then $k+1 \in T$. Then $T = \mathbb{N}$.
- **Second Principle of Mathematical Induction (SPMI2)**.
  Let U be a subset of $\mathbb{N}$. Suppose $0 \in U$. Further suppose that for any $k \in \mathbb{N}$, if $[\![0, k]\!] \subset U$ then $k+1 \in U$. Then $U = \mathbb{N}$.

Proof of Theorem (6). Here we give the set-up and the beginning steps for each part of the argument. Fill in the remaining steps as an exercise.

Argument for '(WOPI)$\Longrightarrow$(SPMI)':

- Suppose the statement (WOPI) holds.

  Let T be a subset of $\mathbb{N}$. Suppose $0 \in T$. Further suppose that for any $n \in \mathbb{N}$, if $k \in T$ then $k+1 \in T$.

  By definition, $T \subset \mathbb{N}$. We verify that $\mathbb{N} \subset T$, with the method of proof by contradiction.

  [*Idea.* Suppose it were true that $\mathbb{N} \not\subset T$. We look for a contradiction.
  By assumption, there would exist some $x_0 \in \mathbb{N}$ such that $x_0 \notin T$.
  Define $S = \{x \in \mathbb{N} : x \notin T\}$. Since $x_0 \in S$, we would have $S \neq \emptyset$.
  (WOPI) would tell us that S had a least element, say, λ. By studying the number λ, we will come to a desired contradiction.
  We will be forced to conclude that $\mathbb{N} \subset T$ in the first place.]

  Since $T \subset \mathbb{N}$ and $\mathbb{N} \subset T$, we have $T = \mathbb{N}$.

  It follows that the statement (SPMI) holds.

Argument for '(SPMI)$\Longrightarrow$(SPMI2)':

- Suppose that the statement (SPMI) holds.

  Let U be a subset of $\mathbb{N}$. Suppose that $0 \in U$. Further suppose that for any $k \in \mathbb{N}$, if $[\![0, k]\!] \subset U$ then $k+1 \in U$.

  By definition, $U \subset \mathbb{N}$. We verify that $\mathbb{N} \subset U$.

  [*Idea.* Define $T = \{x \in \mathbb{N} : [\![0, x]\!] \subset U\}$.
  Apply (SPMI) to deduce that $T = \mathbb{N}$.
  Now ask: Is it true that every element of T an element of U? If yes, then it follows that $\mathbb{N} \subset U$, and furthermore $\mathbb{N} = U$.]

  It follows that the statement (SPMI2) holds.

Argument for '(SPMI2)$\Longrightarrow$(WOPI)':

- Suppose that the statement (SPMI2) holds.

  Let S be a non-empty subset of $\mathbb{N}$. We verify that S has a least element with the method of proof by contradiction:

  [*Idea.* Suppose it were true that S did not have a least element. We look for a contradiction.
  Define $U = \mathbb{N} \backslash S$.
  We verify that $0 \in U$, and that for any $k \in \mathbb{N}$, if $[\![0, k]\!] \subset U$ then $k+1 \in U$.
  (SPMI2) would then imply that $U = \mathbb{N}$. It would then follow that $S = \emptyset$: this is the desired contradiction.]

  It follows that the statement (WOPI) holds.

10. **Further examples of statements which may be proved using mathematical induction.**

    You have learnt the results below in school maths or in the introductory *calculus* course or in the introductory *linear algebra*. They may be proved with the help of mathematical induction. The first step in the argument for each of them is to appropriately formulate the proposition on which Principle of Mathematical Induction is to be applied.

    (a) **Binomial Theorem for numbers.**

    Let a, b be any numbers.
    For any $n \in \mathbb{N}$,

$$\begin{aligned}(a+b)^n &= a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots \\ &\quad + \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n-1} ab^{n-1} + b^n.\end{aligned}$$

    (b) **Leibniz's Rule (on repeated differentiation for products of functions).**

    Suppose g, h are real-valued functions of one real variable which are n-times differentiable at a. Then $g \cdot h$ is n-times differentiable at a and

$$\begin{aligned}(g \cdot h)^{(n)}(a) &= g(a)h^{(n)}(a) + \binom{n}{1} g'(a)h^{(n-1)}(a) + \binom{n}{2} g''(a)h^{(n-2)}(a) + \cdots \\ &\quad + \binom{n}{k} g^{(k)}(a)h^{(n-k)}(a) + \cdots + \binom{n}{n-1} g^{(n-1)}(a)h'(a) + g^{(n)}(a)h(a).\end{aligned}$$

(c) **Remainder Theorem and Factor Theorem 'combined'**

Let $n \in \mathbb{N} \backslash \{0\}$. Let $f(x)$ be a polynomial of degree n. Suppose α is a number. Then there exists some unique polynomial $g(x)$ of degree $n-1$ such that $f(x) = (x-\alpha)g(x) + f(\alpha)$ as polynomials.

Remark. The Remainder Theorem and the Factor Theorem that you learnt in school maths are immediate consequences of this result:

- **Remainder Theorem.**
  Let $f(x)$ be a polynomial. Let α be a number.
  The remainder of $f(x)$ upon division by $x - \alpha$ is $f(\alpha)$.
- **Factor Theorem.**
  Let $f(x)$ be a polynomial. Let α be a number.
  $f(x)$ is divisible by $x - \alpha$ iff $f(\alpha) = 0$.

(d) **Generalized Triangle Inequality (on the complex plane).**

Let $n \in \mathbb{N} \backslash \{0, 1\}$. Suppose $\mu_1, \mu_2, \cdots, \mu_n \in \mathbb{C}$. Then $\left| \sum_{j=1}^{n} \mu_j \right| \leq \sum_{j=1}^{n} |\mu_j|$.

(e) **Existence Theorem for row equivalence to reduced row echelon forms.**

Let p, q be positive integers. Suppose A is a $(p \times q)$-matrix with real entries. Then there exists some $(p \times q)$-matrix with real entries B such that B is a reduced row echelon form and A is row equivalent to B.

Remark. The corresponding 'uniqueness result' can also be proved with the help of mathematical induction.

(f) **Fundamental Theorem of Arithmetic.**

Let $n \in \mathbb{N} \backslash \{0, 1\}$. The statements below hold:

(1) *n is a prime number or a product of several prime numbers.*

(2) *Let $p_1, p_2, \cdots, p_s, q_1, q_2, \cdots, q_t$ be prime numbers. Suppose $0 < p_1 \leq p_2 \leq \cdots \leq p_s$ and $0 < q_1 \leq q_2 \leq \cdots \leq q_t$. Further suppose $n = p_1 p_2 \cdot \ldots \cdot p_s$ and $n = q_1 q_2 \cdot \ldots \cdot q_t$. Then $s = t$ and $p_1 = q_1$, $p_2 = q_2$, ... , $p_s = q_s$.*

Remark. Euclid's Lemma is needed in the argument.