1. **Undefined notions: 'set', 'belong to', 'element'.**

   We leave *unexplained* the meaning of the three statements 'x belongs to the set A', 'x is an element of A', 'A contains x as an element'. At this level we allow heuristics to take over: for us all mean the same thing intuitively: x is an object amongst a collection of objects which is denoted by A. (The latter is also *unexplained.*)

   To save time, we write '$x \in A$' as a short-hand for 'x is an element of A'.

   We write '$x \notin A$' as a short-hand for 'x is not an element of A' (or equivalently 'x does not belong to the set A').

   It may appear strange that something is left unexplained. The 'problem' is that the notions 'set', 'belong to', 'element' are left *undefined*. But these appear to be fundamental in set language. (We are going to define everything else in set language in terms of these notions.)

   This is deliberate. The reason is:

   - To give the definition of an object, or a type of objects, is to specify what this object, or this type of objects, is amongst more general objects, or more general types of objects, which we have understood well enough. In other words, we 'anchor' the definition of what we want to explain via the definition upon something more general but well understood.

     In the situation of 'sets', we can 'anchor' upon nothing which is satisfactory. For example, we may 'define' a set as a collection. But we will then ask what we mean by 'collection'. We may 'define' a collection as an aggregate. But we will then ask what we mean by 'aggregate'. This will stop nowhere.

2. **Heuristic understanding of the notions of 'set equality', 'subset relation'**.

   (a) Any two sets A, B are equal to each other as sets exactly when each of A, B contains as its elements every element of the other. In this situation we write $A = B$.

   (b) A 'relation' between two sets which is somewhat weaker than set equality is 'subset relation':

   - Given any two sets A, B, A is a subset of B exactly when every element of A is an element of B. In this situation we write $A \subset B$. (Some people's convention is '$A \subseteq B$'.)

   Reminder: When $A = B$ holds, it will happen that $A \subset B$ also holds.

3. **'Small' sets**.

   When a set has 'finitely many' elements, we may list every one of them exhaustively. We agree to express such a set simply as a list in which every element of this set is presented, with the symbol '{' signifying the beginning of this list, and the symbol '}' signifying the end of this list.

   Example. Suppose $\star, *, \sharp, \flat, \natural$ are the only elements of a set. This set may be expressed as $\{\star, *, \sharp, \flat, \natural\}$.

   Conventions.

   1. 'Repetition in the list' does not count.

      Provided an object, say, $\star$, is an element of a 'small' set, say, A, $\star$ has to be presented at least once in a list representing A. However, no matter how many more times $\star$ is presented, it still counts as once.

      Example: $\{\sharp, \sharp, \flat, \natural, \flat, \flat\} = \{\sharp, \flat, \natural\}$.

   2. 'Ordering in the list' does not matter.

      Given any two lists, as long every object which is presented in each list is also presented in the other, the two lists will represent the same 'small' set, regardless of the order in which the objects is presented in each list.

      Example: $\{\sharp, \flat, \natural\} = \{\flat, \sharp, \natural\} = \{\natural, \flat, \sharp\}$.

   The set which has no element is called the **empty set**. We denote this set by $\emptyset$.

4. **Method of Specification**.

   Many a set cannot be presented as a list, because it is not 'small'. Even though a set may be presented as an list, for one reason or other we may choose not to do so.

   Examples:

   1. Consider the collection '$0, 1, 4, 9, 16, 25, 36, \cdots$'.

      Is it apparent that it refers to the collection of all square integers?

      But why can't it be understood as the collection of $0, 1, 4, 9, 16, 25$ and the integers no less than 36?

   2. Consider the collection '$1, 2, 3, 4, \cdots; \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots; \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \cdots; ...$'

      Is it apparent that it refers to the collection of all positive rational numbers? Or is it not?

      Can you conceive a better list than this one?

      Or is it desirable to describe the collection of all positive rational numbers in this way?

When it is impossible or undesirable to present a set by exhaustively listing every element of the set, we may try the **Method of Specification**.

In such a set presented witht the Method of Specification, its elements are those objects, and those alone, which turn a predicate 'used for describing that set' into a true statement.

Recall: A **predicate with variables** $x, y, z, \cdots$ is a statement 'modulo' the ambiguity of possibly one or several variables $x, y, z, \cdots$. Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

Suppose A is a set, and $P(x)$ is a predicate with variable x.

1. $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x for which the statement $P(x)$ is true.
2. $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x which is an element of the given set A and for which the statement $P(x)$ is true. By definition it is a subset of A.

5. **Examples on Method of Specification**.

(a) $\{x \mid x = \star \text{ or } x = * \text{ or } x = \sharp \text{ or } x = \flat \text{ or } x = \natural\}$ is the same as $\{\star, *, \sharp, \flat, \natural\}$ as sets.

(b) What is the set $\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}$ in plain words?
This is the solution set of the equation $x^2 - 3x + 2 = 0$ with unknown x in $\mathbb{R}$. It is the set $\{1, 2\}$.

(c) What are the sets below in plain words?
  (i) $\{x \in \mathbb{R} : x^2 + 1 = 0\}$.
  This is the solution set of the equation $x^2 + 1 = 0$ with unknown x in $\mathbb{R}$. It is the empty set.
  (ii) $\{x \in \mathbb{C} : x^2 + 1 = 0\}$.
  This is the solution set of the equation $x^2 + 1 = 0$ with unknown x in $\mathbb{C}$. It is $\{i, -i\}$.

Remark. But how about $\{x \mid x^2 + 1 = 0\}$?
The answer depends on the context of discussion. Are we talking about a set of real numbers? Or a set of complex numbers? Or a set of 2×2-matrix with real entries? (Or complex entries)?

(d) Heuristically describe the set $\{x \in \mathbb{R} : \sqrt{x} \in \mathbb{N}\}$.
$\{x \in \mathbb{R} : \sqrt{x} \in \mathbb{N}\}$ is the same as $\{0, 1, 4, 9, 16, \cdots, n^2, (n+1)^2, \cdots\}$. (We have eliminated the annoying dots.)
Remark. How to see the answer? Ask: If $x \in \mathbb{R}$ and $\sqrt{x} \in \mathbb{N}$, what can x be? Now collect all such x's.

(d') What about these sets below?
  (i) $\{x \in \mathbb{R} : \text{There exists some } n \in \mathbb{N} \text{ such that } x = n^2\}$.
  (ii) $\{x \in \mathbb{R} : x = n^2 \text{ for some } n \in \mathbb{N}\}$.
  (iii) $\{x \mid x = n^2 \text{ for some } n \in \mathbb{N}\}$.

Each of them is the same as $\{0, 1, 4, 9, 16, \cdots, n^2, (n+1)^2, \cdots\}$. Because of (iii), we also accept this set to be expressed as $\{n^2 \mid n \in \mathbb{N}\}$.

(d") What about $\{x \in \mathbb{Z} : \text{There exists some } n \in \mathbb{N} \text{ such that } x = n + \sqrt[3]{n}\}$?
This is the same as $\{0, 2, \cdots, m^3 + m, (m+1)^3 + m + 1, \cdots\}$.

(d"') What about $\{x \in \mathbb{R} : x = n^2 \text{ for any } n \in \mathbb{N}\}$?
This is the empty set. (Why? Inspect the predicate '$x = n^2$ for any $n \in \mathbb{N}$'.)

(e) Heuristically describe the set $\{x \mid x = 3^m 5^n \text{ for some } m, n \in \mathbb{N}\}$.
This is the same as $\{1, 3, 9, 27, \cdots; 5, 15, 45, \cdots; 25, 75, 225, \cdots; ...\}$.
Remark. We also accept this set to be expressed as $\{3^m 5^n \mid m \in \mathbb{N} \text{ and } n \in \mathbb{N}\}$.

(f) How to apply the Method of Specification to express the collection $0, 3, 6, \cdots, 3n, 3n+3, \cdots$ as a set? There are many correct answers. Each of them refer to the same set.

- $\{3n \mid n \in \mathbb{N}\}$,
- $\{x \mid x = 3n \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{Z} : x = 3n \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{N} : x/3 \in \mathbb{N}\}$.

(f') How about the collection $-6, -3, 0, 3, 6, \cdots$?
There are many correct answers. Each of them refer to the same set.

- $\{3(n-2) \mid n \in \mathbb{N}\}$,
- $\{x \mid x = 3(n-2) \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{Z} : x = 3(n-2) \text{ for some } n \in \mathbb{N}\}$,
- $\{x \in \mathbb{Z} : (x+6)/3 \in \mathbb{N}\}$.

(f") How about the collection $\cdots, -6, -3, 0, 3, 6, \cdots$?
There are many correct answers. Each of them refer to the same set.

- $\{3n \mid n \in \mathbb{Z}\}$,
- $\{x \mid x = 3n \text{ for some } n \in \mathbb{Z}\}$,
- $\{x \in \mathbb{Z} : x = 3n \text{ for some } n \in \mathbb{Z}\}$,
- $\{x \in \mathbb{Z} : x/3 \in \mathbb{Z}\}$.

(g) When there are many solutions for a given equation, the method of specification may be useful in the presentation of all solutions in the form of a 'solution set'.
What is the set of all real solutions of the equation $\sin(x) = 0$ with unknown x?

- $\{n\pi \mid n \in \mathbb{Z}\}$,
- $\{x \in \mathbb{R} : x = n\pi \text{ for some } n \in \mathbb{Z}\}$.

Remark. We are actually asking for the 'general solution of the equation $\sin(x) = 0$ with unknown x in $\mathbb{R}$'. When we give the answer as '$x = n\pi$ *where n is an arbitrary integer*', what we actually mean is: '*$x = \alpha$ is a real solution of this equation in $\mathbb{R}$ iff ($\alpha = n\pi$ for some $n \in \mathbb{Z}$).*'

(g') What is the set of all real solutions of the equation $\sin(x) = \frac{1}{2}$ with unknown x?

- $\left\{ n\pi + (-1)^n \cdot \frac{\pi}{6} \;\middle|\; n \in \mathbb{Z} \right\}$,
- $\left\{ x \in \mathbb{R} : x = n\pi + (-1)^n \cdot \frac{\pi}{6} \text{ for some } n \in \mathbb{Z} \right\}$.

(h) What is the set of all real solutions of the system of equation

$$(S) : \left\{ \begin{array}{rcrcrcr} x_1 & - & 5x_2 & + & 3x_3 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & = & -1 \end{array} \right.$$

with unknown x_1, x_2, x_3 in $\mathbb{R}$?

- $\left\{ \begin{bmatrix} -2/3 + (7/6)t \\ -1/3 + (5/6)t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$,
- $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{array}{l} \text{There exists some } t \in \mathbb{R} \text{ such that} \\ x_1 = -\frac{2}{3} + \frac{7}{6}t \text{ and } x_2 = -\frac{1}{3} + \frac{5}{6}t \text{ and } x_3 = t \end{array} \right\}$.

Remark. What we are saying, without using the jargon of set language, is that '$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *is a solution of the system (S) iff there exists some $t \in \mathbb{R}$ such that $x_1 = -\frac{2}{3} + \frac{7}{6}t$ and $x_2 = -\frac{1}{3} + \frac{5}{6}t$ and $x_3 = t$.*'

(h') The method of specification is used extensively in constructions in *linear algebra*.
Below are the simplest examples:

- Let H be a $(m \times n)$-matrix with real entries.
  The null space of H is $\{\mathbf{x} \in \mathbb{R}^n : H\mathbf{x} = \mathbf{0}\}$.
- Let H be a $(m \times n)$-matrix with real entries.
  The column space of H is $\left\{ \mathbf{y} \in \mathbb{R}^m : \begin{array}{l} \text{There exists some } \mathbf{x} \in \mathbb{R}^n \\ \text{such that } \mathbf{y} = H\mathbf{x} \end{array} \right\}$.
- Let V be a subspace of $\mathbb{R}^n$.
  The orthogonal complement of V is $\{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for any } \mathbf{x} \in V\}$.

Each of them can be generalized in a natural way with the help of notion of linear transformation.

(i) What is the set $\{x \mid x \neq x\}$?
This is the empty set.
Reason?
Warning. We can formally construct, using the method of specification, the objects

$$\{x \mid x = x\}, \qquad \{x \mid x \notin x\}$$

We would expect these objects to be sets. However, it will turn out that they cannot be 'reasonably regarded as sets', if we are to insist that all sets are to obey certain laws which look natural and which govern their behaviour.

6. **Definitions of the basic set operations, with the help of the Method of Specification.**
Let A, B be sets.

(a) The **intersection** of the sets A, B is defined to be the set $\{x \mid x \in A \text{ and } x \in B\}$.

(b) The **union** of the sets A, B is defined to be the set $\{x \mid x \in A \text{ or } x \in B\}$.

(c) The **complement of the set B in the set A** is defined to be the set $\{x \mid x \in A \text{ and } x \notin B\}$.

(d) The **symmetric difference** of the sets A, B is defined to be the set $(A \backslash B) \cup (B \backslash A)$.

They are denoted by $A \cap B$, $A \cup B$, $A \backslash B$, $A \triangle B$ respectively.

Before we state the basic properties of these set operations, we need explain the notion of 'set equality' and 'subset relation' by giving some appropriate definitions.

7. **Formal definition of the notions of 'set equality', 'subset relation'**.

Recall that any two sets A, B are equal to each other as sets exactly when each of A, B contains as its elements every element of the other. A more clumsy way to write 'each of A, B contains the same objects as its elements as the other does' is: every element of A is an element of B and every element of B is an element of A. Hence a formal definition for the notion of 'set equality' is:

- Let A, B be sets. A is said to be **equal** to B if both of the following statements (†), (‡) hold:
  - (†) For any object x, [if $(x \in A)$ then $(x \in B)$].
  - (‡) For any object y, [if $(y \in B)$ then $(y \in A)$].

  We write $A = B$.

Formal and clumsy though it looks, it is safest to proceed from this definition when we are doing calculations or giving some proofs.

As for the notion of 'subset relation', a formal definition is:

- Let A, B be sets. A is said to be a **subset** of B if the following statement (†) holds:
  - (†) For any object x, [if $(x \in A)$ then $(x \in B)$].

  We write $A \subset B$ (or $B \supset A$).

8. **Properties of the basic set operations**.

Theorem (I). *The following statements hold:*

(1) *Let A be a set. $A \subset A$.*

(2) *Let A, B be sets. $A = B$ iff [$(A \subset B)$ and $(B \subset A)$].*

(3) *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Then $A \subset C$.*

Theorem (II). *Let A, B be sets. The following statements hold:*

(1) $A \cap B \subset A$.

(2) $A \cap B \subset B$.

(3) $A \backslash B \subset A$.

(4) $A \subset A \cup B$.

(5) $B \subset A \cup B$.

Theorem (III). *Let A be a set. The following statements hold:*

(1) $\emptyset \subset A$.

(2) $A \cap \emptyset = \emptyset$.

(3) $A \cup \emptyset = A$.

(4) $A \backslash \emptyset = A$.

(5) $\emptyset \backslash A = \emptyset$.

(6) $A \triangle \emptyset = A$.

(7) $A \triangle A = \emptyset$.

Theorem (IV). *The following statements hold:*

(1) *Let A, B, S be sets. Suppose $S \subset A$ and $S \subset B$. Then $S \subset A \cap B$.*

(2) *Let A, B, S be sets. Suppose $S \subset A$ or $S \subset B$. Then $S \subset A \cup B$.*

(3) *Let A, B, T be sets. Suppose $A \subset T$ and $B \subset T$. Then $A \cup B \subset T$.*

(4) *Let A, B, T be sets. Suppose $A \subset T$ or $B \subset T$. Then $A \cap B \subset T$.*

Theorem (V). *Let A, B, C be sets. The following statements hold:*

(1) $A \cap A = A$.

(2) $A \cap B = B \cap A$.

(3) $(A \cap B) \cap C = A \cap (B \cap C)$.

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(5) $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$.

(6) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

(7) $A \triangle B = (A \cup B) \backslash (A \cap B)$.

(8) $A \triangle B = B \triangle A$.

(9) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(1') $A \cup A = A$.

(2') $A \cup B = B \cup A$.

(3') $(A \cup B) \cup C = A \cup (B \cup C)$.

(4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(5') $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.

(6') $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.