1. In school mathematics we have tacitly accepted that it makes sense to talk about complex numbers.

   Each such number is a mathematical object of the form $a + bi$, in which a, b are some real numbers, and i is a 'special number' satisfying the relation $i^2 = -1$.

   'Understood' at school:

   - *Let a, b, c, d be real numbers. Consider the complex numbers $a + bi$, $c + di$.*

   (a) *We define the sum of $a + bi$ and $c + di$ as $(a + c) + (b + d)i$.*

   (b) *We define the difference of $a + bi$ from $c + di$ as $(a - c) + (b - d)i$.*

   (c) *We define the product of $a + bi$ and $c + di$ as $(ac - bd) + (ad + bc)i$.*

   (d) *Provided that $c \neq 0$ or $d \neq 0$, we define the quotient of $a + bi$ by $c + di$ as* $\dfrac{ac + bd}{c^2 + d^2} + \dfrac{-ad + bc}{c^2 + d^2} i$.

   We tacitly assumed that these 'arithmetic operations' obey the 'usual laws of arithmetic' which hold for real numbers.

   We will work on these assumptions for now, and also add into the mix what we have learnt in plane coordinate geometry.

2. **Definition.**

Let z be a complex number.

Denote the real part and the imaginary part of z by $\mathsf{Re}(z)$, $\mathsf{Im}(z)$ respectively.

(So $z = \mathsf{Re}(z) + i\mathsf{Im}(z)$.)

- *The* **complex conjugate** *of z is defined to be the complex number*

$$\mathsf{Re}(z) - i\mathsf{Im}(z).$$

  It is denoted by $\overline{z}$.

- *The* **modulus** *of z, denoted by $|z|$, is defined to be the non-negative real number*

$$\sqrt{(\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2}.$$

3. **Theorem (1)**.

Let z, w be complex numbers. The statements below hold:

(1) $\mathsf{Re}(z) = \dfrac{z+\overline{z}}{2}$ *and* $\mathsf{Im}(z) = \dfrac{z-\overline{z}}{2i}$.

(2) $\mathsf{Re}(\overline{z}) = \mathsf{Re}(z)$,

$\mathsf{Im}(\overline{z}) = -\mathsf{Im}(z)$ *and* $|\overline{z}| = |z|$.

(3) $\overline{\overline{z}} = z$.

(4) $\mathsf{Re}(z+w) = \mathsf{Re}(z) + \mathsf{Re}(w)$ *and*

$\mathsf{Im}(z+w) = \mathsf{Im}(z) + \mathsf{Im}(w)$.

(5) $\mathsf{Re}(zw) = \mathsf{Re}(z)\mathsf{Re}(w) - \mathsf{Im}(z)\mathsf{Im}(w)$

and

$\mathsf{Im}(zw) = \mathsf{Re}(z)\mathsf{Im}(w) + \mathsf{Im}(z)\mathsf{Re}(w)$.

(6) $\overline{z+w} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\,\overline{w}$.

(7) $|z|^2 = z\overline{z}$.

(8) $|zw| = |z||w|$.

(9) $|\mathsf{Re}(z)| \leq |z|$ *and* $|\mathsf{Im}(z)| \leq |z|$.

(10) $|\mathsf{Re}(z\overline{w})| \leq |z||w|$.

(11) $|z+w| \leq |z| + |w|$ *and* $||z| - |w|| \leq |z-w|$.

(12) $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

Proof. Exercise. (Straightforward manipulation.)

4. **Complex numbers and Euclidean plane geometry.**

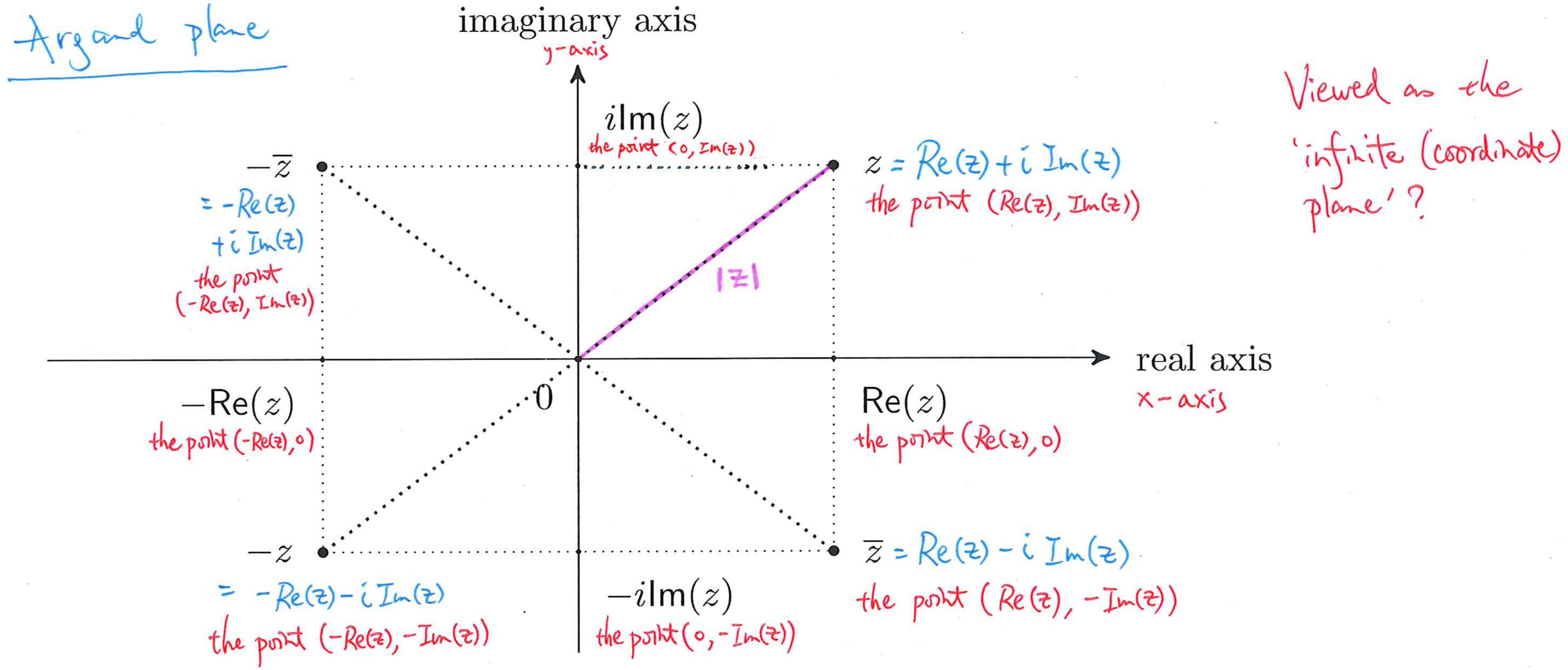
Identify each complex number z as the point $(\mathsf{Re}(z), \mathsf{Im}(z))$ on the 'infinite plane'.

We refer to the point $(\mathsf{Re}(z), \mathsf{Im}(z))$ simply as 'the point z'.

In this context we call the 'infinte plane' as the **Argand plane**.

$|z|$ is the (Euclidean) distance between 0 and z.

$\overline{z}$ is the point obtained by reflecting the point z with respect to the 'real axis'.



Remark.

The basic definitions and results concerned with the arithmetic of complex numbers, together with all results in Theorem (1), can be given geometric interpretation on the Argand plane.

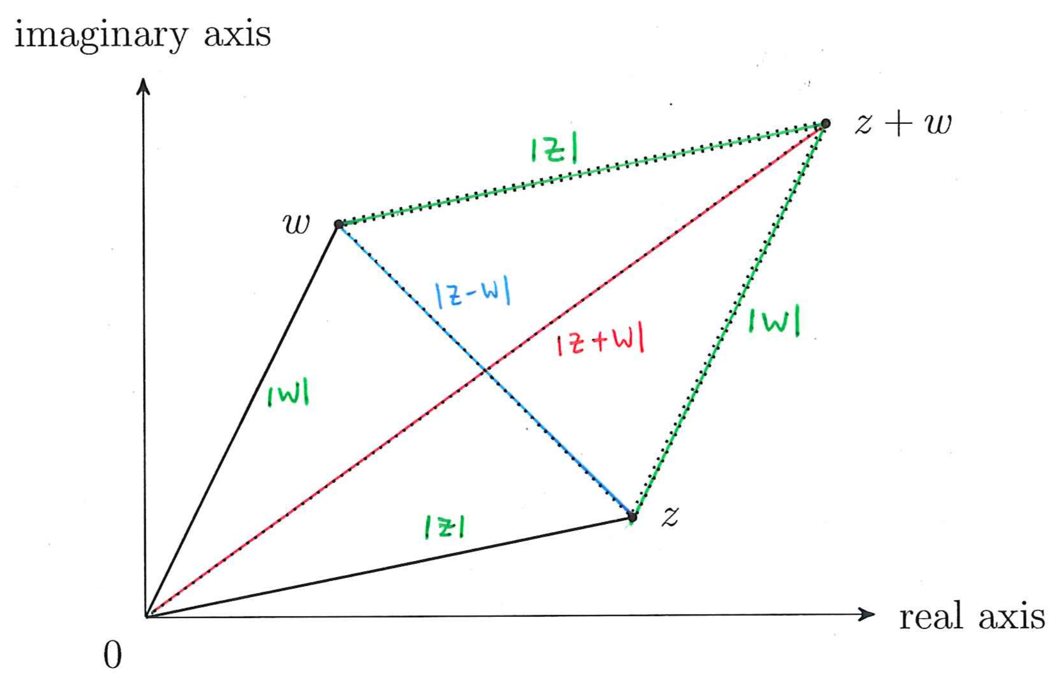
Further remark.

The basic definitions, such as collinearity, perpendicularity and parallelism, and results in Euclidean plane geometry can be expressed in terms of complex numbers.

5. **Theorem (2). (Geometric interpretation of addition.)**

Let z, w be distinct non-zero complex numbers. The points 0, z, w, $z + w$ are the vertices of a parallelogram in which:

- *the line segment joining 0 and z is parallel to the line segment joining w and $z+w$,*
- *the line segment joining 0 and w is parallel to the line segment joining z and $z+w$, and*
- *the line segment joining z and w and the line segment joining 0 and $z + w$ are diagonals.*



Remarks.

(a) Hence the relation $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$ is known as the paralellogramic identity in this context.

(b) We may further interpret 'adding z by w' as: translating the line segment joinint 0 to z to the line segment joining w and $z+w$ via parallelism.

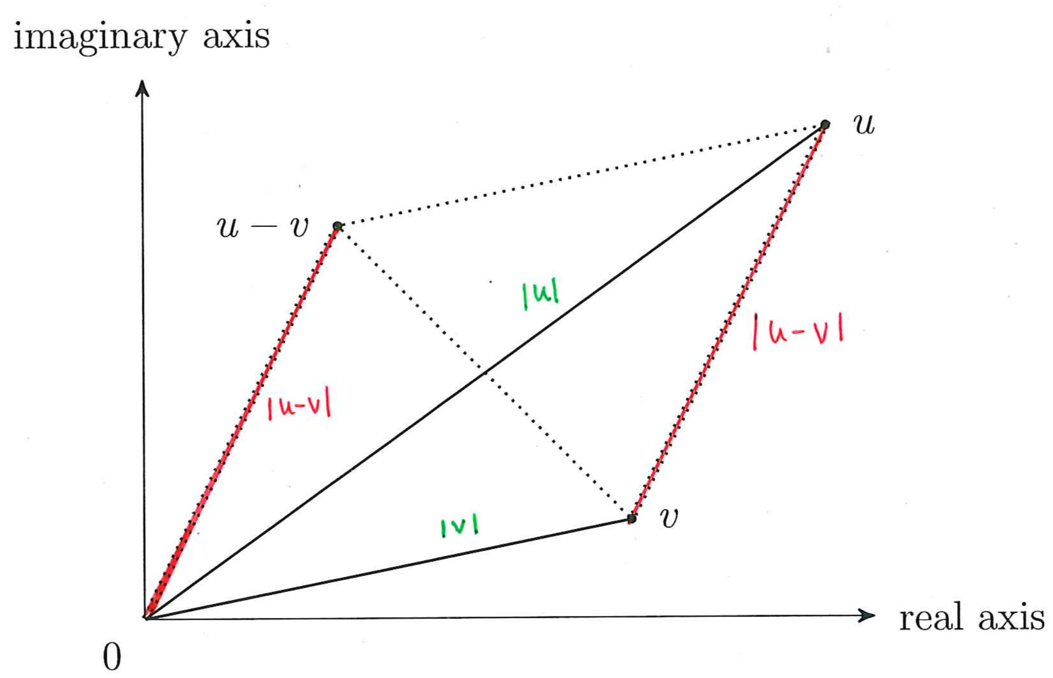
(c) Can you provide a geometric interpretation for the inequality $|z+w| \leq |z| + |w|$?

This is known as the Triangle Inequality

6. **Corollary to Theorem (1). (Geometric interpretation of subtraction.)**

Let u, v be distinct non-zero complex numbers. The point $u - v$ is the number so that 0, v, $u - v$, v forms the vertices of a parallelogram in which:

- *the line segment joining 0 and $u - v$ is parallel to the line segment joining v and u,*
- *the line segment joining 0 and v is parallel to the line segment joining $u - v$ and u, and*
- *the line segment joining 0 and u and the line segment joining v and $u - v$ are diagonals.*



Remarks.

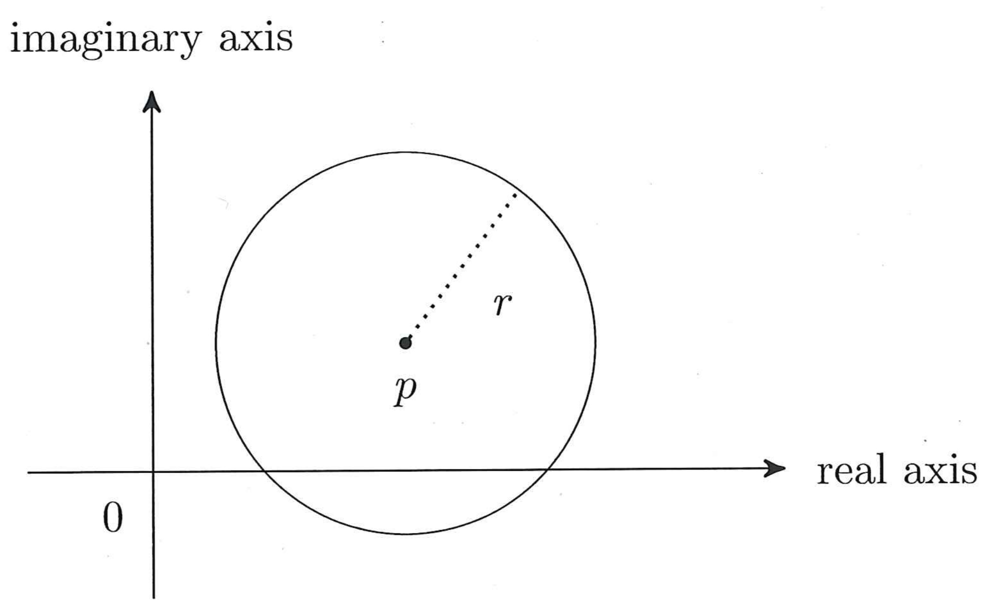
(a) We may further interpret 'subtracting u by v' as: translating the line segment joinint v to u to the line segment joining 0 and $u - v$ via parallelism.

(b) $|u - v|$ is the (Euclidean) distance between u and v.

7. **Theorem (3). (Circles, and the geometric interpretation of modulus.)**

Let p be a complex number, and r be a positive number.

The equation $|z-p| = r$ with unknown z in the complex numbers describes the circle with centre p and of radius r.



Remark. We may think of the equation $|z - p| = 0$ as the 'degenerate' circle with centre p and of radius 0.

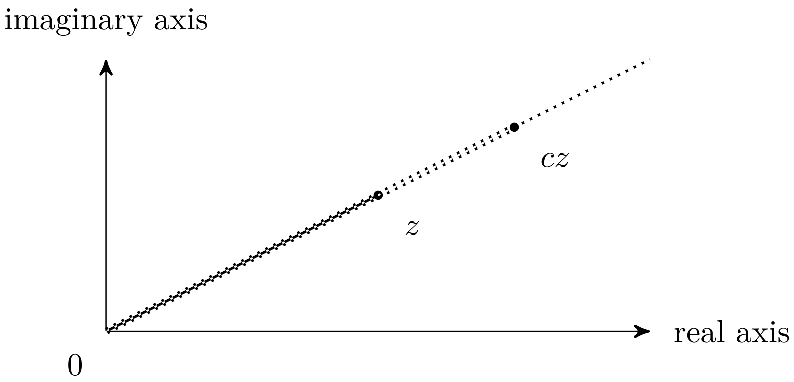
Further remark. How about the geometric interpretation of the inequality '$|z-p| < r$' with unknown z?

8. **Theorem (4). (Geometric interpretation of multiplication by real numbers.)**

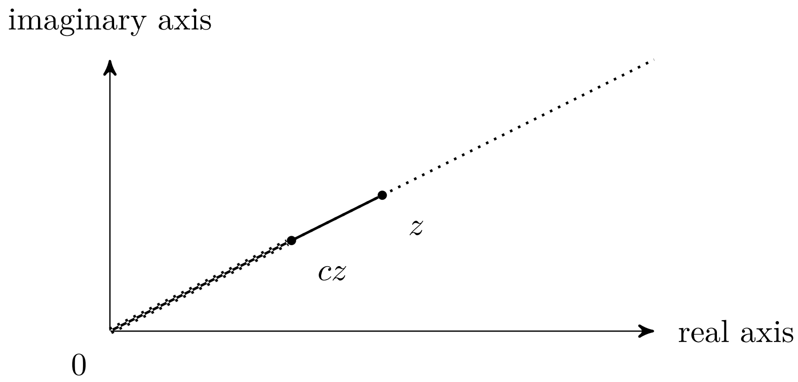
Let z be a non-zero complex number, and c be a real number.

(a) *Suppose $c > 0$. Then cz is the complex number on the same half line starting from 0 and joining z so that the distance between 0 and cz is $c|z|$.*

Picture in the scenario '$c > 1$'



Picture in the scenario '$0 < c < 1$'



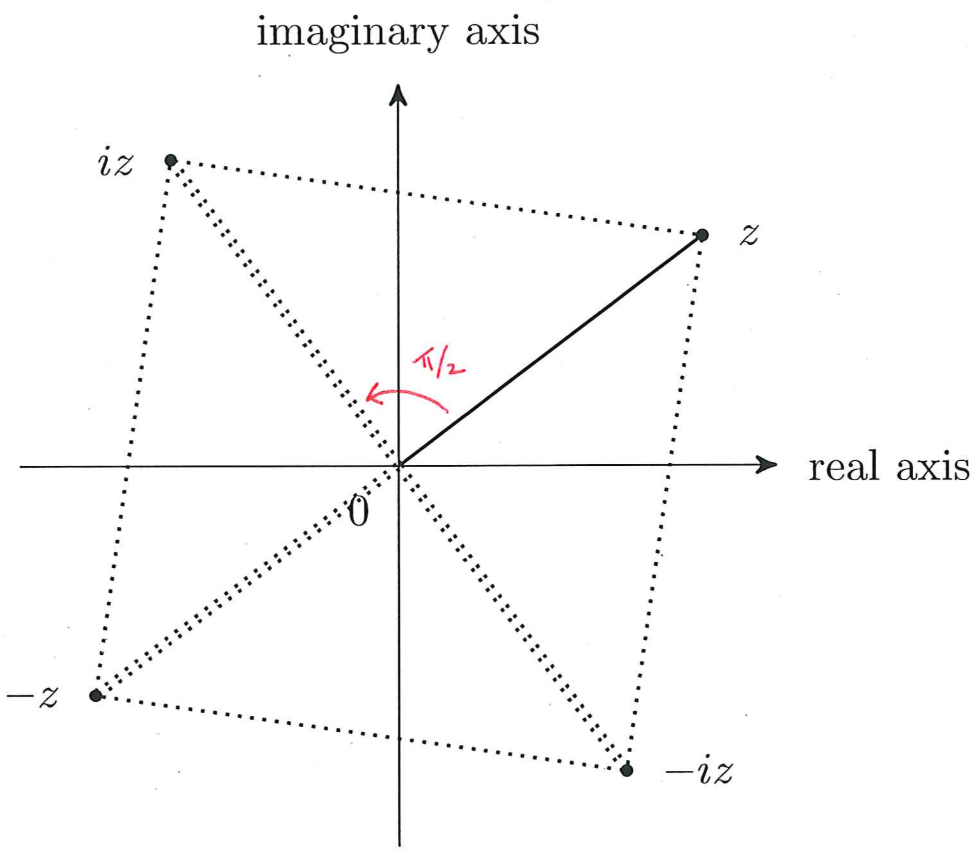
(b) *Suppose $c < 0$. Then cz is the complex number on the same half line starting from 0 and joining $-z$ so that the distance between 0 and cz is $c|z|$.*

9. **Theorem (5). (Geometric interpretation of multiplication by i)**

Suppose z be a non-zero complex number.

Then the points z, iz, $-z$, $-iz$ are the four vertices of a square in which the line segment joining $z, -z$ and the line segment joining $iz, -iz$ are the diagonals.

The latter is obtained by rotating the former about the point 0 by $\pi/2$ radians, so that iz is obtained from z, and $-iz$ is obtained from $-z$.



10. **Theorem (6).**

Let z be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.

Remark.

(a) The expression $z = \mathsf{Re}(z) + i\mathsf{Im}(z)$ is referred to as the **standard form** of z.

(b) The expression $z = |z|(\cos(\theta) + i\sin(\theta))$ (for some appropriate θ) is referred to as the **polar form** of z.

10. **Theorem (6).**

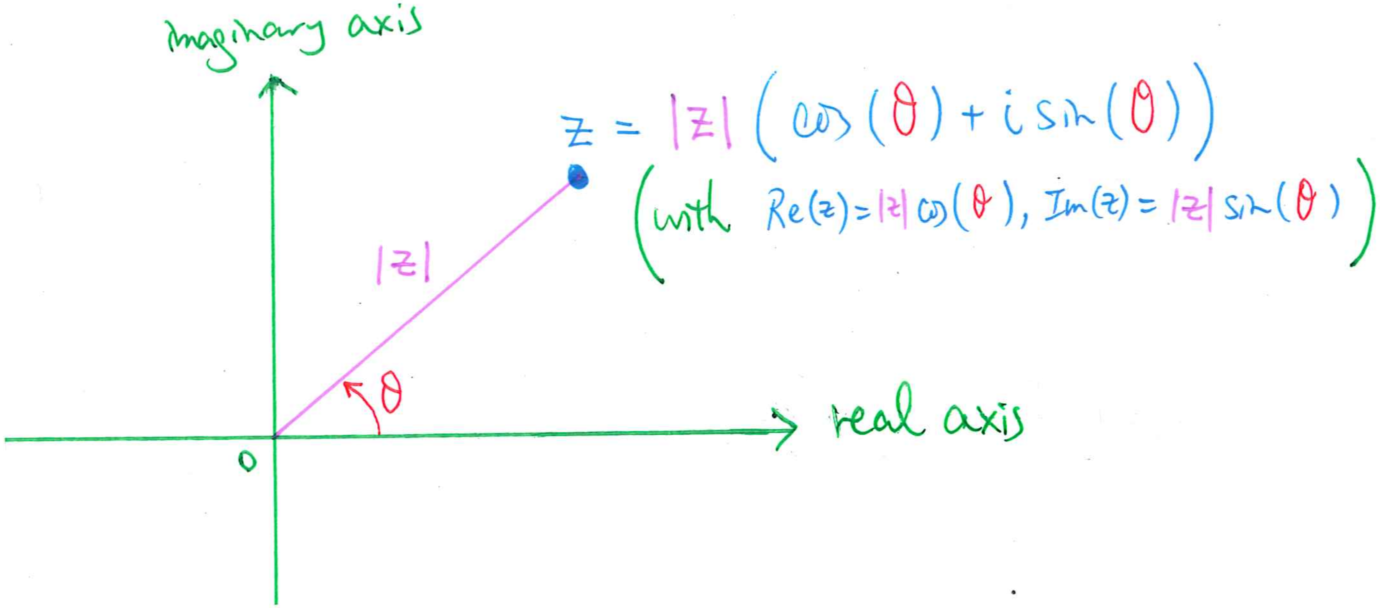
Let z be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta)+i\sin(\theta))$.

Remark.

(a) The expression $z = \mathsf{Re}(z) + i\mathsf{Im}(z)$ is referred to as the **standard form** of z.

(b) The expression $z = |z|(\cos(\theta) + i\sin(\theta))$ (for some appropriate θ) is referred to as the **polar form** of z.

- What does Theorem (6) tell us? It tells us that for each complex number z, we have such a picture:



Starting from '$z = |z|(\cos(\theta)+i\sin(\theta))$', we have $\begin{cases} Re(z) = |z|\cos(\theta) \\ Im(z) = |z|\sin(\theta) \end{cases}$

Starting from '$z = Re(z) + i\,Im(z)$', we have $|z| = \sqrt{(Re(z))^2 + (Im(z))^2}$.

But can we name a θ satisfying $\begin{cases} Re(z) = |z|\cos(\theta) \\ Im(z) = |z|\sin(\theta) \end{cases}$?

Theorem (6).

Let z be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.

Proof. Let z be a complex number. We have $z = 0$ or ($z \neq 0$ and $\mathsf{Re}(z) \geq 0$) or ($z \neq 0$ and $\mathsf{Re}(z) < 0$).

(Case 1). Suppose $z = 0$.

Then $|z| = 0$.

Therefore $z = 0 = |z|(\cos(0) + i\sin(0))$.

(Case 2). Suppose $z \neq 0$ and $Re(z) \geq 0$.

Note that $-1 \leq \frac{Im(z)}{|z|} \leq 1$.

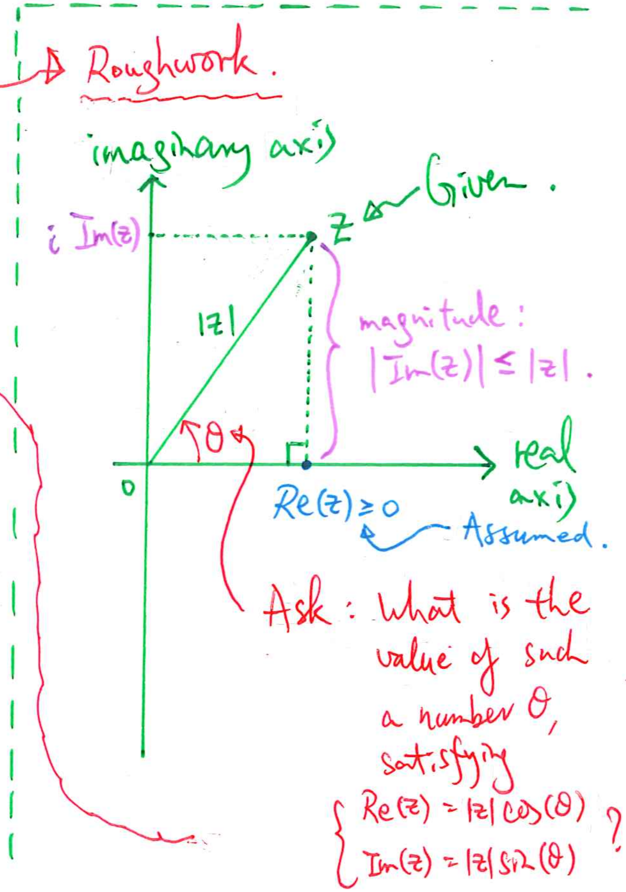
Define $\theta = \arcsin\left(\frac{Im(z)}{|z|}\right)$.

We have $Im(z) = |z|\sin(\theta)$ by definition.

Also, $Re(z) = \sqrt{|z|^2 - (Im(z))^2} = \ldots = |z|\cos(\theta)$.

Then $z = Re(z) + i\,Im(z)$

$= \ldots = |z|(\cos(\theta) + i\sin(\theta))$.



Theorem (6).

Let z be a complex number. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.

Proof. Let z be a complex number. We have $z = 0$ or ($z \neq 0$ and $\mathsf{Re}(z) \geq 0$) or ($z \neq 0$ and $\mathsf{Re}(z) < 0$).

(Case 1). ...

(Case 2). ...

(Case 3). Suppose $z \neq 0$ and $Re(z) < 0$.

Define $w = -z$. We have

$w \neq 0$ and $Re(w) = Re(-z) = -Re(z) > 0$.

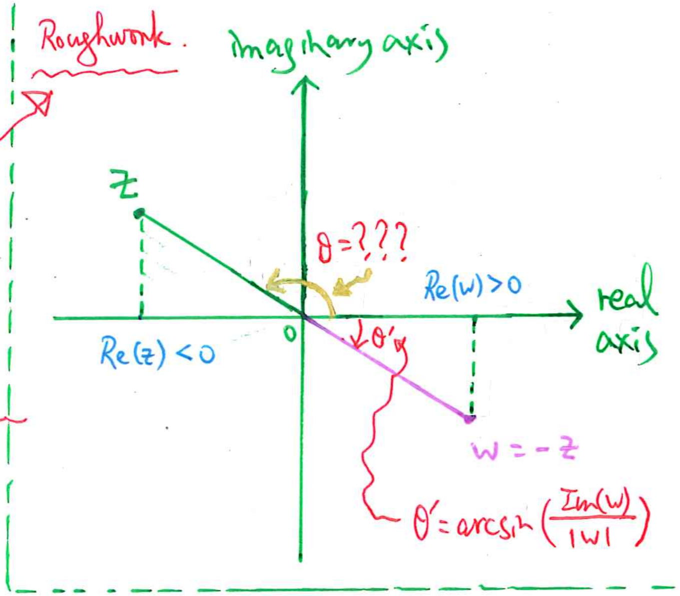
By the argument in (Case 2),

there exists some $\theta' \in \mathbb{R}$, namely, $\theta' = \arcsin\left(\frac{Im(w)}{|w|}\right)$

such that $w = |w|(\cos(\theta') + i\sin(\theta'))$.

Define $\theta = \theta' + \pi$.

Then $z = -w = -|w|(\cos(\theta') + i\sin(\theta')) = \ldots = |z|(\cos(\theta) + i\sin(\theta))$. □



11. **Definition.**

Let z be a non-zero complex number, and θ be a real number.

Suppose the equality $z = |z|(\cos(\theta)+i\sin(\theta))$ holds. Then θ is said to be an **argument** *for z.*

Further suppose $-\pi < \theta \leq \pi$. Then θ is called the **principal argument** *for z, and we write $\theta = \arg(z)$.*

Remark.

- We do not write '*the* argument of the (non-zero) complex number so-and-so', because the same non-zero complex number has 'infinitely many' different arguments. (For example, for each $n \in \mathbb{Z}$, $2n\pi$ is an argument of 1.)
- However, each non-zero complex number has exactly one principal argument. So we should write '*the* principal argument of the (non-zero) complex number so-and-so'.

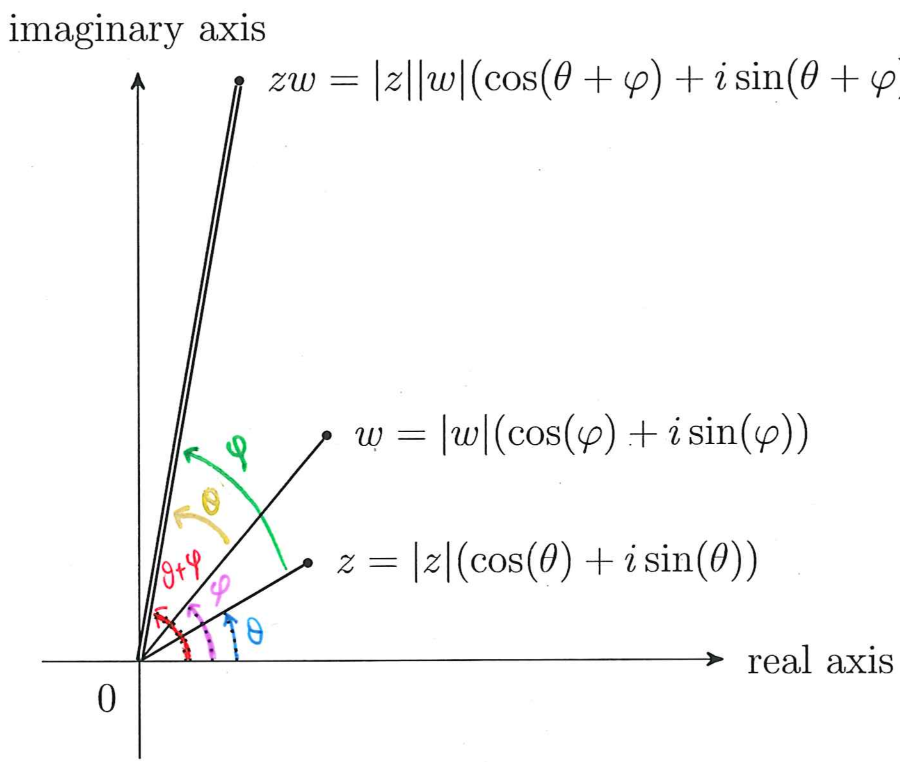
12. **Theorem (7). (Multiplication and division for complex numbers in polar form.)**

Suppose z, w are non-zero complex numbers, with arguments θ, φ respectively. Then:

(a) $zw = |z||w|(\cos(\theta+\varphi) + i\sin(\theta+\varphi))$, *and* $\dfrac{z}{w} = \dfrac{|z|}{|w|}(\cos(\theta-\varphi) + i\sin(\theta-\varphi))$.

(b) *The modulus of zw is $|z||w|$, and the modulus of z/w is $|z|/|w|$.*

(c) *$\theta+\varphi$ is an argument for zw, and $\theta-\varphi$ is an argument for $\dfrac{z}{w}$.*



13. **Corollary to Theorem (7).**

Let z be a non-zero complex number, and θ be an argument of z.

$z^2 = |z|^2(\cos(2\theta) + i\sin(2\theta))$.

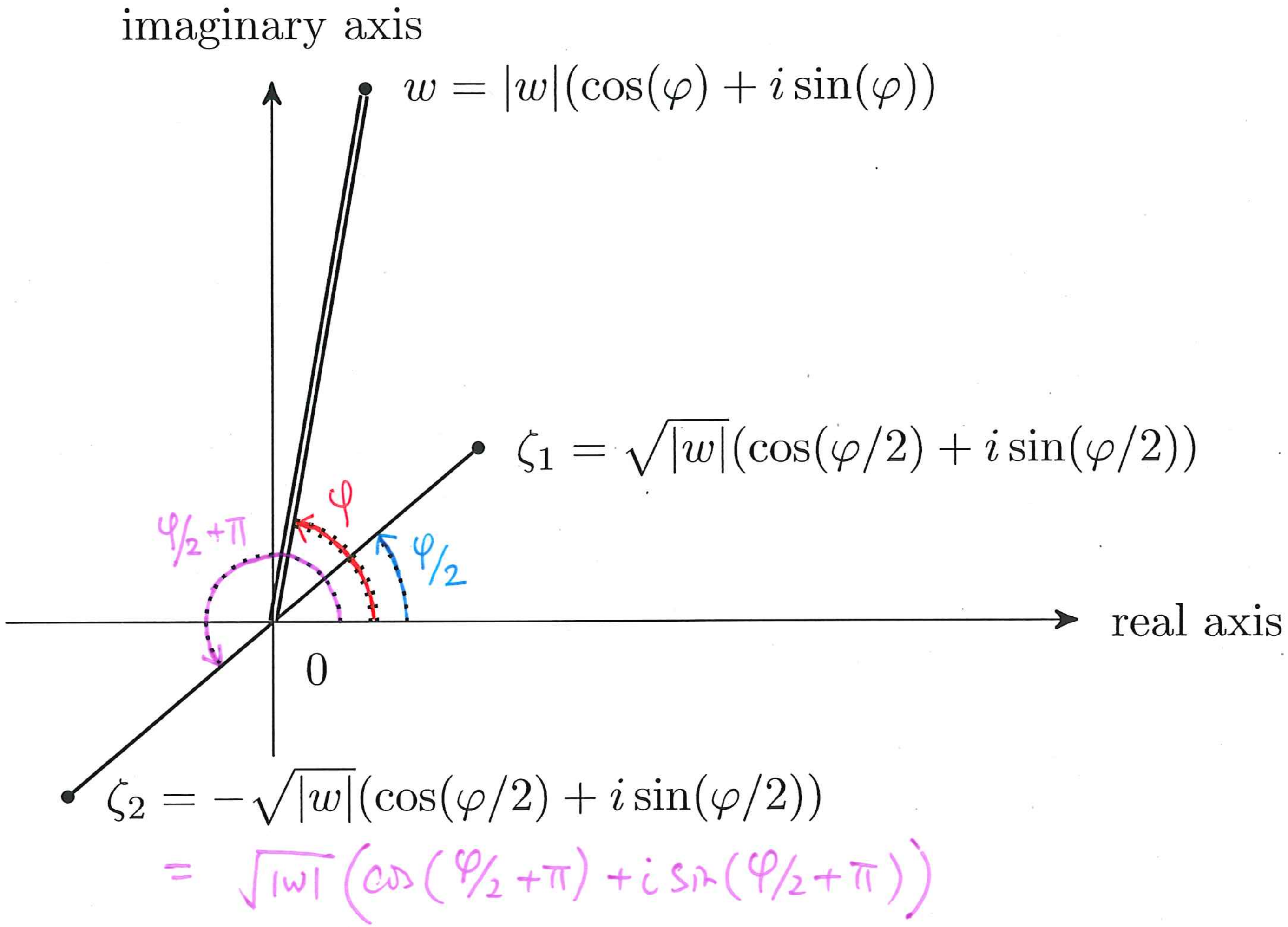
14. **Theorem (8).**

Let w, ζ be non-zero complex numbers, and φ be an argument of w. Suppose $w = \zeta^2$.

Then $\zeta = \sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right)$ *or* $\zeta = -\sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right)$.

Remark. The two distinct non-zero complex numbers $\sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right)$, $-\sqrt{|w|}\left(\cos\left(\frac{\varphi}{2}\right) + i\sin\left(\frac{\varphi}{2}\right)\right)$ are the **square roots** of the complex number w.

Further remark. How to interpret Theorem (8) in terms of plane geometry?



15. **Theorem (9). (Roots of quadratic polynomials with complex coefficients.)**

Let a, b, c be complex numbers, with $a \neq 0$. Let α be a number. Let $f(z)$ be the quadratic polynomial given by $f(z) = az^2 + bz + c$.

(a) *Suppose α is a root of $f(z)$. Let $\beta = -b/a - \alpha$. Then the statements below hold:*

i. $f(z)=a(z-\alpha)(z-\beta)$ *as polynomials.*

ii. β *is a root of* $f(z)$.

iii. $\alpha\beta = c/a$.

(b) *Define $\Delta_f = b^2 - 4ac$. We call Δ_f the discriminant of the polynomial $f(z)$. Then the statements below hold:*

i. $f(z) = a\left[\left(z + \frac{b}{2a}\right)^2 - \frac{\Delta_f}{4a^2}\right]$ *as polynomials.*

ii. *Suppose $\Delta_f \neq 0$. Suppose σ is a square root of $\Delta_f/(4a^2)$. Define $\alpha_\pm = -b/2a \pm \sigma$ respectively. Then $f(z)$ has two distinct roots amongst the complex numbers, namely α_+, α_-, and $f(z)$ is factorized as $f(z) = a(z - \alpha_+)(z - \alpha_-)$.*

iii. *Now suppose $\Delta_f = 0$ instead. Then $f(z)$ has a repeated root, namely, $-b/2a$, amongst the complex numbers, and $f(z)$ is factorized as $f(z) = a(z + b/2a)^2$.*

Remark.

What the above result says is that each quadratic polynomial with complex coefficients $f(z)$ has a pair of roots and 'factorizes into linear polynomials'.

Moreover, if the polynomial $f(z)$ is given by $f(z) = az^2 + bz + c$ and the pair of roots concerned are α, β, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Furthermore, regarding the quadratic equation

$$az^2 + bz + c = 0 \quad \text{———} \quad (\star)$$

with unknown x, there are exactly two mutually exclusive possibilities:

(1) Suppose $\Delta_f \neq 0$. Then the equation $(\star)$ has exactly two distinct solutions amongst the complex numbers.

(2) Suppose $\Delta_f = 0$. Then the equation $(\star)$ has exactly one repeated solution amongst the complex numbers.

In any case, the equation $(\star)$ has at least one solution amongst the complex numbers.

16. **Appendix 1: Complex numbers and polynomials.**

Theorem (9) is significant in two ways:

(a) This result for quadratic polynomials is a 'baby case' of the **Fundamental Theorem of Algebra**, first proved by Gauss, which says:

- *Every non-constant polynomial with coefficients in complex numbers has at least one root amongst the complex numbers.*

In fact Gauss gave several proofs for this result.

(b) We can express all the roots of every quadratic polynomial with coefficients in complex numbers in terms of its coefficients with the help of the operations $+, -, \times, \div$, and with the taking of (square) roots.

So it is natural to ask whether we can do the same thing for cubic polynomials, quartic polynomials, quintic polynomials et cetera.

Answer:

Yes for cubic polynomials and quartic polynomials.

No in general for higher-degree polynomials.

17. **Appendix 2: linear algebra done with complex numbers.**

Everything in your beginning *linear algebra* course which involves real numbers and addition, subtraction, multiplication and division for real numbers alone can be adapted to complex numbers.

(a) Gaussian elimination can be adapted for solving systems of linear equations with complex given's and complex unknowns.

(b) General results on systems of linear equations with real given's and real unknowns can be adapted as results on systems of linear equations with complex given's and complex unknowns.

(c) The rules of operations which apply to matrices, column vectors, and determinants with real entries can be adapted to matrices, column vectors and determinants with complex entries.

(d) The notions of row operations, reduced row-echelon forms, non-singularity, and invertibility, matrix inverses can adapted.

(e) Subspaces of $\mathbb{C}^n$ (over $\mathbb{C}$) are defined in the same manner as subspaces of $\mathbb{R}^n$ (over $\mathbb{R}$):

- *Suppose* V *is a non-empty subset of* $\mathbb{C}^n$*. Then* V *is said to form a subspace of* $\mathbb{C}^n$ *over* $\mathbb{C}$ *if, for any* $\alpha, \beta \in \mathbb{C}$*, for any* $\mathbf{x}, \mathbf{y} \in V$, $\alpha\mathbf{x} + \beta\mathbf{y} \in V$.

(f) The notions of null space, column space can be adapted by changing the word 'real' to the word 'complex' in the respective definitions.

Let A *be an* $(m \times n)$*-matrix with complex entries.*

i. *The null space of* A *is defined to be the set* $\{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\}$.

ii. *The column space of* A *is defined to be the set* $\left\{ \mathbf{y} \in \mathbb{C}^m : \begin{array}{l} \textit{There exists some } \mathbf{x} \in \mathbb{C}^n \\ \textit{such that } \mathbf{y} = A\mathbf{x} \end{array} \right\}$.

(g) The notions of linear combination, span, linear dependence/independence, basis can be adapted by changing the word 'real' to the word 'complex' in the respective definitions.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k, \mathbf{v} \in \mathbb{C}^n$.

i. $\mathbf{v}$ *is said to be a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *if there exist some* $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{C}$ *such that* $\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k$.

ii. *The span of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *is defined to be the set* $\left\{ \mathbf{v} \in \mathbb{C}^n : \begin{array}{l} \mathbf{v} \textit{ is a linear combination} \\ \textit{of } \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k. \end{array} \right\}$.

iii. $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *are said to be linear dependent over* $\mathbb{C}$ *if there exist some* $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{C}$, *not all zero, such that* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$.

iv. $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *are said to be linear independent over* $\mathbb{C}$ *if for any* $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{C}$, *if* $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k = \mathbf{0}$ *then* $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

v. *Let* V *be a subspace of* $\mathbb{C}^n$. *Suppose* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k \in V$.

$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *is said to constitute a basis for* V *if the span of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *is* V *and* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ *are linear independent over* $\mathbb{C}$.

All results formulated in terms of these concepts can be adapted by changing the word 'real' to the word 'complex' in the statements for the results.