MATH1050 Arithmetic progression and geometric progression

1. **Definition.**
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ is said to be an **arithmetic progression** *if the statement (AP) holds:*

(AP) *There exists some $d \in \mathbb{C}$ such that for any $n \in \mathbb{N}$, $a_{n+1} - a_n = d$.*

The number d is called the **common difference** *of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$.*

Remark. The use of the article '*the*' should be justified with a proof for this statement:

Each arithmetic progression has at most one common difference.

What is it, really? In a less 'compact' (but more clumsy) form, this reads:

'*Let $\{a_n\}_{n=0}^{\infty}$ be an arithmetic progression, and d, d' be complex numbers. Suppose d, d' are common differences of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$. Then $d = d'$.*'

Remark on terminology. Let $N \in \mathbb{N}\backslash\{0,1\}$. Suppose $c_0, c_1, \cdots, c_N$ are $N+1$ complex numbers. We abuse notation in saying that $c_0, c_1, \cdots, c_N$ form an arithmetic progression with common difference d exactly when there exists some arithmetic progression $\{a_n\}_{n=0}^{\infty}$ with common difference d such that $a_k = c_k$ for any integer k amongst $0, 1, 2, \cdots, N$. (In plain language, $c_0, c_1, \cdots, c_N$ are identified as the 0-th term, 1-st term, ... , N-th term of the arithmetic progression $\{a_n\}_{n=0}^{\infty}$.)

2. **Lemma (1).**
Suppose $\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression, with common difference d.
Then $a_{m+n} = a_m + nd$ for any $m, n \in \mathbb{N}$. (In particular $a_n = a_0 + nd$ for any $n \in \mathbb{N}$.)

3. **Theorem (2). (Equivalent formulations of the definition of arithmetic progression.)**
Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. The statements below are logically equivalent:
   (a) *$\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression.*
   (b) *There exists some $d \in \mathbb{C}$ such that for any $n \in \mathbb{N}$, $a_n = a_0 + nd$.*
   (c) *For any $k \in \mathbb{N}$, $a_{k+2} - a_{k+1} = a_{k+1} - a_k$.*
   (d) *For any $k \in \mathbb{N}$, $a_{k+1} = \dfrac{a_k + a_{k+2}}{2}$.*
   (e) *For any $k \in \mathbb{N}$, the numbers a_k, a_{k+1}, a_{k+2} form an arithmetic progression.*

4. **Lemma (3).**
Let $n \in \mathbb{N}$. The equality $0 + 1 + 2 + 3 + \cdots + (n-1) + n = \dfrac{n(n+1)}{2}$ holds.

Remark on its proof. Write '$0 + 1 + 2 + 3 + \cdots + (n-1) + n$' as '$n + (n-1) + (n-2) + (n-3) + \cdots + 1 + 0$'. Add the latter with the former and then count how many copies of n is obtained in the sum.

5. **Theorem (4). (Sum of arithmetic progression.)**
Let $\{a_n\}_{n=0}^{\infty}$ be an arithmetic progression with common difference d.
For any $m, n \in \mathbb{N}$, $a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = (n+1)a_m + \dfrac{n(n+1)}{2}d$.

6. **Definition.**
Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}\backslash\{0\}$. The infinite sequence $\{b_n\}_{n=0}^{\infty}$ is said to be a **geometric progression** *if the statement (GP) holds:*

(GP) *There exists some $r \in \mathbb{C}\backslash\{0\}$ such that for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$.*

The number r is called the **common ratio** *of this geometric progression.*

Remark. The use of the article '*the*' should be justified with a proof for this statement:

Each geometric progression has at most one common ratio.

What is it, really? In a less 'compact' (but more clumsy) form, this reads:

'*Let $\{b_n\}_{n=0}^{\infty}$ be a geometric progression, and r, r' be complex numbers. Suppose r, r' are common ratios of the geometric progression $\{b_n\}_{n=0}^{\infty}$. Then $r = r'$.*'

Remark on terminology. Let $N \in \mathbb{N}\backslash\{0,1\}$. Suppose $c_0, c_1, \cdots, c_N$ are $N+1$ non-zero complex numbers. We abuse notation in saying that $c_0, c_1, \cdots, c_N$ form a geometric progression with common ratio r exactly when there exists some geometric progression $\{b_n\}_{n=0}^{\infty}$ with common ratio r such that $b_k = c_k$ for any integer k amongst $0, 1, 2, \cdots, N$. (In plain language, $c_0, c_1, \cdots, c_N$ are identified as the 0-th term, 1-st term, ... , N-th term of the geometric progression $\{b_n\}_{n=0}^{\infty}$.)

7. **Lemma (5).**

   Suppose $\{b_n\}_{n=0}^{\infty}$ is a geometric progression, with common ratio r.

   Then $b_{m+n} = b_m r^n$ for any $m, n \in \mathbb{N}$. (In particular $b_n = b_0 r^n$ for any $n \in \mathbb{N}$.)

8. **Theorem (6). (Equivalent formulations of the definition of geometric progression.)**

   Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}\backslash\{0\}$. The statements below are logically equivalent:

   (a) *$\{b_n\}_{n=0}^{\infty}$ is a geometric progression.*

   (b) *There exists some $r \in \mathbb{C}\backslash\{0\}$ such that for any $n \in \mathbb{N}$, $b_n = b_0 r^n$.*

   (c) *For any $k \in \mathbb{N}$, $\dfrac{b_{k+2}}{b_{k+1}} = \dfrac{b_{k+1}}{b_k}$.*

   (d) *For any $k \in \mathbb{N}$, $b_{k+1}^2 = b_k b_{k+2}$.*

   (e) *For any $k \in \mathbb{N}$, the numbers b_k, b_{k+1}, b_{k+2} form a geometric progression.*

9. **Lemma (7).**

   Let $n \in \mathbb{N}$ and $r \in \mathbb{C}$. The statements below hold:

   (a) $1 - r^{n+1} = (1-r)(1 + r + r^2 + \cdots + r^n)$.

   (b) *Further suppose $r \neq 1$. Then* $\dfrac{1 - r^{n+1}}{1-r} = 1 + r + r^2 + \cdots + r^n$.

   Remark on its proof. Multiply $1 + r + r^2 + \cdots + r^n$ by r to obtain $r + r^2 + r^3 + \cdots + r^{n+1}$. Subtract the latter from the former and see what happens.

10. **Theorem (8). (A useful 'identity'.)**

    Let $n \in \mathbb{N}$. Let $s, t \in \mathbb{C}$. The equality $s^{n+1} - t^{n+1} = (s-t)(s^n + s^{n-1}t + s^{n-2}t^2 + \cdots + s^{n-k}t^k + \cdots + st^{n-1} + t^n)$ holds.

    Remark on its proof. Apply Lemma (7): 'formally' substitute $r = t/s$ and multiply both sides by s^{n+1}.

    Remark. When we want to 'factorizing' the expression $s^n - t^n$ or $s^n + t^n$ with the help of integers only, we have these equalities below for 'small values' of n:

    $$\begin{aligned} s^2 - t^2 &= (s-t)(s+t), \\ s^3 - t^3 &= (s-t)(s^2 + st + t^2), & s^3 + t^3 &= (s+t)(s^2 - st + t^2), \\ s^4 - t^4 &= (s-t)(s+t)(s^2 + t^2), \\ s^5 - t^5 &= (s-t)(s^4 + s^3t + s^2t^2 + st^3 + t^4), & s^5 + t^5 &= (s+t)(s^4 - s^3t + s^2t^2 - st^3 + t^4), \\ s^6 - t^6 &= (s-t)(s+t)(s^2 + st + t^2)(s^2 - st + t^2), & s^6 + t^6 &= (s^2 + t^2)(s^4 - s^2t^2 + t^2), \end{aligned}$$

    Of course, when we resort to roots of unity, we may 'completely factorize' $s^n - t^n$ into a product of 'linear expressions' with the help of complex numbers. Or we may 'factorize' $s^n - t^n$ into a product of 'linear expressions' and 'quadratic expressions' with the help of real numbers only. (We need De Moivre's Theorem and results about roots of unity here.)

11. **Theorem (9). (Sum of geometric progression.)**

    Let $\{b_n\}_{n=0}^{\infty}$ be a geometric progression with common ratio r. Then for each $m, n \in \mathbb{N}$,

    $$b_m + b_{m+1} + b_{m+2} + \cdots + b_{m+n} = \begin{cases} (n+1)b_m & \text{if } \; r = 1 \\ \dfrac{b_m(r^{n+1} - 1)}{r-1} & \text{if } \; r \neq 1 \end{cases}$$

12. **Theorem (10). (Limiting value for a sum of geometric progression.)**

    Let $\{b_n\}_{n=0}^{\infty}$ be a geometric progression with common ratio r. Suppose $|r| < 1$.

    Then $\lim\limits_{n\to\infty} (b_0 + b_1 + b_2 + \cdots + b_n) = \dfrac{b_0}{1-r}$.

    Remark on its proof. Apply Lemma (7), and standard techniques in *calculus* such as the Sandwich Rule.

    Further remark. In school mathematics, you were concerned with the situation in which r was real. But this result holds even when r is not a real number.