1. **Convention on the summation symbol.**
   Suppose m, n are integers.
   (a) *Suppose $m \leq n$. Suppose $a_m, a_{m+1}, a_{m+2}, \cdots, a_{n-1}, a_n$ are numbers.*
   We agree to write $a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$ as $\sum_{k=m}^{n} a_k$.
   (b) *Suppose $m > n$. Then we agree to read $\sum_{k=m}^{n} b_k$ as 0 no matter what the b_k's are.*

   Remark. The symbol k in the expression $\sum_{k=m}^{n} a_k$ is called the dummy index in this expression.

2. **Theorem (1). (Basic properties of the summation symbol.)**
   Let m, n, p, q be integers. Suppose $m \leq n \leq p$. Let $a_m, a_{m+1}, \cdots, a_n, b_m, b_{m+1}, \cdots, b_n, c$ be numbers. The statements below hold:
   (a) $\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_j$.
   (b) $\sum_{k=m}^{m} a_k = a_m$.
   (c) $\sum_{k=m}^{n} a_k = \sum_{j=m-q}^{n-q} a_{j+q}$.
   (d) $\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_{m+n-j}$.
   (e) $\sum_{k=m}^{n} c = (n-m+1)c$.
   (f) $\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k$.
   (g) $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$.
   (h) $\sum_{k=m}^{p} a_k = \sum_{k=m}^{n} a_k + \sum_{k=n+1}^{p} a_k$.

   Remark. Do not be scared by the symbols. Many of these formulae are just 'short-hand' for what you are very familiar. For instance:

   - '$\sum_{k=m}^{n} a_k = \sum_{j=m}^{n} a_{m+n-j}$' is '$a_m + a_{m+1} + \cdots + a_{n-1} + a_n = a_n + a_{n-1} + \cdots + a_{m+1} + a_m$' in disguise.
   - '$\sum_{k=m}^{n} ca_k = c\sum_{k=m}^{n} a_k$' is '$ca_m + ca_{m+1} + \cdots ca_n = c(a_m + a_{m+1} + \cdots + a_n)$' in disguise.

   Proof. Apply mathematical induction. (Hence they are postponed at this moment but will be left to you).

3. **Some basic results on 'double summation'.**
   The results stated below on 'double summation' are left as exercises. These results can be generalized to 'triple summation' et cetera.
   Theorem (2).
   Let m, n, p, q be integers. Suppose $m \leq n$ and $p \leq q$. Suppose

$$\begin{matrix} a_{m,p}, & a_{m,p+1}, & \cdots & a_{m,q}, \\ a_{m+1,p}, & a_{m+1,p+1}, & \cdots & a_{m+1,q}, \\ \vdots & \vdots & & \vdots \\ a_{n,p}, & a_{n,p+1}, & \cdots & a_{n,q} \end{matrix}$$

   are numbers. Then $\sum_{j=m}^{n}\sum_{k=p}^{q} a_{j,k} = \sum_{k=p}^{q}\sum_{j=m}^{n} a_{j,k}$.
   Corollary (3).
   Let m, n be integers. Suppose $m \leq n$. Suppose

$$\begin{matrix} a_{m,m}, & a_{m,m+1}, & a_{m,m+2} & \cdots & a_{m,n-1} & a_{m,n}, \\ & a_{m+1,m+1}, & a_{m+1,m+2} & \cdots & a_{m+1,n-1} & a_{m+1,n}, \\ & & a_{m+2,m+2} & \cdots & a_{m+2,n-1} & a_{m+2,n}, \\ & & & \ddots & \vdots & \vdots \\ & & & & a_{n-1,n-1}, & a_{n-1,n}, \\ & & & & & a_{n,n} \end{matrix}$$

   are numbers. Then $\sum_{j=m}^{n}\sum_{k=j+1}^{n} a_{j,k} = \sum_{k=m}^{n}\sum_{j=m}^{k} a_{j,k}$.

Theorem (4).

Let m, n, p, q be integers. Suppose $m \leq n$ and $p \leq q$. Suppose $a_m, a_{m+1}, a_{m+2}, \cdots, a_{n-1}, a_n, b_p, b_{p+1}, b_{p+2}, \cdots, b_{q-1}, b_q$ are numbers.

Then $\left(\sum_{j=m}^{n} a_j\right)\left(\sum_{k=p}^{q} b_k\right) = \sum_{j=m}^{n}\sum_{k=p}^{q} a_j b_k = \sum_{k=p}^{q}\sum_{j=m}^{n} a_j b_k.$

4. **Convention on the product symbol.**

   Suppose m, n are integers.

   (a) *Suppose $m \leq n$. Suppose $a_m, a_{m+1}, a_{m+2}, \cdots, a_{n-1}, a_n$ are numbers.*

   We agree to write $a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \ldots \cdot a_{n-1} \cdot a_n$ as $\prod_{k=m}^{n} a_k$.

   (b) *Suppose $m > n$. Then we agree to read $\prod_{k=m}^{n} b_k$ as 1 no matter what the b_k's are.*

   Remark. The symbol k in the expression $\prod_{k=m}^{n} a_k$ is called the dummy index in this expression.

5. **Theorem (5). (Basic properties of the product symbol.)**

   Let m, n, p, q be integers. Suppose $m \leq n \leq p$. Let $a_m, a_{m+1}, \cdots, a_n, b_m, b_{m+1}, \cdots, b_n, c$ be numbers. The statements below hold:

   (a) $\prod_{k=m}^{n} a_k = \prod_{j=m}^{n} a_j.$

   (b) $\prod_{k=m}^{m} a_k = a_m.$

   (c) $\prod_{k=m}^{n} a_k = \prod_{j=m-q}^{n-q} a_{j+q}.$

   (d) $\prod_{k=m}^{n} a_k = \prod_{j=m}^{n} a_{m+n-j}.$

   (e) $\prod_{k=m}^{n} c = c^{n-m+1}.$

   (f) $\prod_{k=m}^{n} ca_k = c^{n-m+1} \prod_{k=m}^{n} a_k.$

   (g) $\prod_{k=m}^{n}(a_k b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right).$

   (h) $\prod_{k=m}^{p} a_k = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=n+1}^{p} a_k\right).$

   Remark. Do not be scared by the symbols. Many of these formulae are just 'short-hand' for what you are very familiar. For instance:

   - '$\prod_{k=m}^{n}(a_k b_k) = \left(\prod_{k=m}^{n} a_k\right)\left(\prod_{k=m}^{n} b_k\right)$' is '$a_m b_m \cdot a_{m+1} b_{m+1} \cdot \ldots \cdot a_n b_n = (a_m \cdot a_{m+1} \cdot \ldots \cdot a_n)(b_m \cdot b_{m+1} \cdot \ldots \cdot b_n)$' in disguise.

   Proof. Apply mathematical induction. (Hence they are postponed at this moment but will be left to you).

6. **Telescopic method for sums and products.**

   Suppose we are given some 'finite sequence' of numbers $b_m, b_{m+1}, b_{m+2}, \cdots, b_n$, and we want to compute the sum $\sum_{k=m}^{n} b_k$. If it happens that there are some appropriate numbers $a_m, a_{m+1}, a_{m+2}, \cdots, a_n, a_{n+1}$ for which $b_j = a_{j+1} - a_j$ holds for each j, then we obtain the equality

   $$\sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$$

   immediately. This method for computing/simplifying sums is referred to as the **telescopic method for sums**.

   There is an analogous **telescopic method for products**; formulate it as an exercise.

7. **Theorem (6). (Mechanism behind the Telescopic Method.)**

   Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers.

   (a) *Whenever $m \leq n$, the equality $\sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$ holds.*

   (b) *Further suppose $a_j \neq 0$ for each j. Then whenever $m \leq n$, the equality $\prod_{k=m}^{n} \frac{a_{k+1}}{a_k} = \frac{a_{n+1}}{a_m}$ holds.*

   Proof. Apply mathematical induction. (Hence they are postponed at this moment but will be left to you).