1. **Absolute value for the reals.**

Definition. (Absolute value of a real number.)

Let r be a real number.

The **absolute value** *of r, which is denoted by $|r|$, is the non-negative real number defined by*

$$|r| = \begin{cases} r & \text{if} \quad r \geq 0 \\ -r & \text{if} \quad r < 0 \end{cases} .$$

Remarks.

(a) In a less formal manner we may refer to $|r|$ is the **magnitude** of the real number r.

(b) This is the geometric interpretation of the definition: $|r|$ is the distance between the point identified as 0 and the point identified as r on the real line.

Lemma (1).

Let $r \in \mathbb{R}$. The statements below hold:

(a) $r \geq 0$ *iff* $|r| = r$.

(b) $r \leq 0$ *iff* $|r| = -r$.

(c) $r = 0$ *iff* $|r| = 0$.

(d) $-|r| \leq r \leq |r|$.

Proof. Exercise in word game on the definition and the word *iff.*

Lemma (2).

Let $r \in \mathbb{R}$. The statements below hold:

(a) $|r|^2 = r^2$.

(b) $|r| = \sqrt{r^2}$.

Remark. What is the relevance of this result? We give one example: whenever we obtain in a calculation the expression $|$'blah-blah-blah'$|^2$, we may replace it by the expression ('blah-blah-blah'$)^2$, which may be easier to handle.

Proof. Let $r \in \mathbb{R}$.

(a) We have $r \geq 0$ or $r < 0$.

(Case 1.) Suppose $r \geq 0$. Then $|r| = r$. Therefore $|r|^2 = r^2$.

(Case 2.) Suppose $r < 0$. Then $|r| = -r$. Therefore $|r|^2 = (-r)^2 = r^2$.

Hence, in any case, $|r|^2 = r^2$.

(b) We have verified that $|r|^2 = r^2$. Since $|r| \geq 0$, we have $|r| = \sqrt{|r|^2} = \sqrt{r^2}$.

Lemma (3).

Let $s, t \in \mathbb{R}$. The equality $|st| = |s||t|$ holds.

Proof. Let $s, t \in \mathbb{R}$. We have $|st|^2 = (st)^2 = s^2t^2 = |s|^2|t|^2 = (|s||t|)^2$. Then $|st| = |s||t|$. (Why?)

Lemma (4).

Let $r, c \in \mathbb{R}$. Suppose $c \geq 0$. Then the statements below hold:

(a) $|r| \leq c$ *iff* $-c \leq r \leq c$.

(b) $|r| < c$ *iff* $-c < r < c$.

(c) $|r| \geq c$ *iff* ($r \leq -c$ *or* $r \geq c$).

(d) $|r| > c$ *iff* ($r < -c$ *or* $r > c$).

Proof. Exercise.

Definition. (Absolute value function.)

The function from $\mathbb{R}$ to $\mathbb{R}$ defined by assigning each real number to its absolute value is called the **absolute value function**.

Remark.

In symbols we may denote this function by $|\cdot|$, and express its 'formula of definition' as '$x \longmapsto |x|$ for each $x \in \mathbb{R}$', or equivalently as

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

We may also express the 'formula of definition' of the function $|\cdot|$ as '$|x| = \sqrt{x^2}$ for any $x \in \mathbb{R}$'.

2. **Triangle Inequality on the real line.**

Lemma (5).

Suppose x, y are real numbers. Then $x^2 + y^2 \geq 2xy$. Equality holds iff $x = y$.

Proof. Suppose x, y are real numbers. We have $(x^2 + y^2) - 2xy = (x - y)^2$.

(a) Since x, y are real, $x - y$ is real. Then $(x - y)^2 \geq 0$. Therefore $x^2 + y^2 \geq 2xy$.

(b) i. Suppose $x = y$. Then $(x^2 + y^2) - 2xy = (x - y)^2 = (x - x)^2 = 0$. Therefore $x^2 + y^2 = 2xy$.

ii. Suppose $x^2 + y^2 = 2xy$. Then $0 = (x^2 + y^2) - 2xy = (x - y)^2$. Therefore $x - y = 0$. Hence $x = y$.

Theorem (6). (Triangle Inequality on the real line.)

Suppose u, v are real numbers. Then $|u + v| \leq |u| + |v|$. Equality holds iff $uv \geq 0$.

Proof.

Suppose u, v are real numbers.

Then $(|u| + |v|)^2 - |u + v|^2 = (|u| + |v|)^2 - (u + v)^2 = (|u|^2 + |v|^2 + 2|u||v|) - (u^2 + v^2 + 2uv) = 2(|uv| - uv)$. (Why?)

(a) We have $uv \leq |uv|$. Then $(|u| + |v|)^2 - |u + v|^2 \geq 0$.
Therefore $|u + v|^2 \leq (|u| + |v|)^2$.
Since $|u + v| \geq 0$ and $|u| + |v| \geq 0$, we have $|u + v| \leq |u| + |v|$.

(b) i. Suppose $uv \geq 0$. Then $|uv| = uv$. Therefore $(|u| + |v|)^2 - |u + v|^2 = 0$. Hence $|u + v| = |u| + |v|$.

ii. Suppose $|u + v| = |u| + |v|$. Then $(|u| + |v|)^2 - |u + v|^2 = 0$. Therefore $|uv| = uv$. Hence $uv \geq 0$.

Remark.

An alternative argument for this result starts in this way:

> *Suppose u, v are real numbers. Then u, v are both non-negative, or u, v are both non-positive, or (one of u, v is non-negative and the other is non-positive).*

Now argue 'case by case'.

Corollary (7). (Corollary to Triangle Inequality on the real line.)

Suppose s, t are real numbers. Then $|\ |s| - |t|\ | \leq |s - t|$. Equality holds iff $st \geq 0$.

3. **Appendix: Triangle Inequality on the plane.**

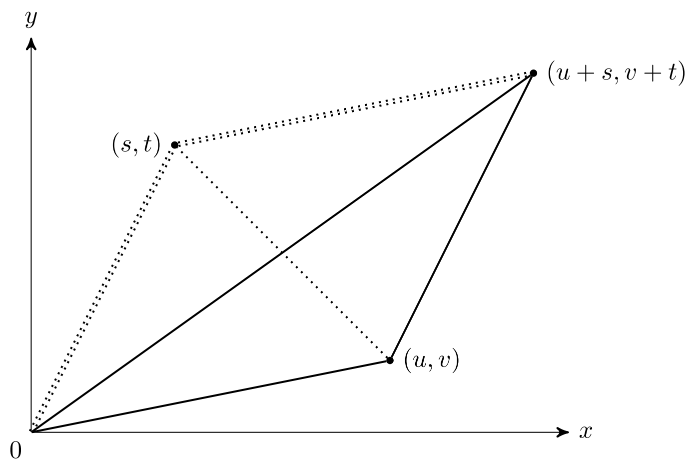
Theorem (6) can be regarded as a special case of Theorem (8). (There are 'higher-dimensional analogues' of this result.)

Theorem (8). (Triangle Inequality on the plane.)

Suppose u, v, s, t are real numbers. Then $\sqrt{(u + s)^2 + (v + t)^2} \leq \sqrt{u^2 + v^2} + \sqrt{s^2 + t^2}$. Equality holds iff ($ut = vs$ and $us \geq 0$ and $vt \geq 0$).

Remark. This is the geometric interpretation of Theorem (8) on the coordinate plane:

> Consider the parallelogram whose vertices are $(0, 0)$, (u, v), (s, t), $(u + s, v + t)$.
> The line segment joining $(0, 0)$, (u, v) has length $\sqrt{u^2 + v^2}$. The line segment joining (u, v), $(u + s, v + t)$, which is the same as the distance between $(0, 0)$, (s, t), has length $\sqrt{s^2 + t^2}$. The line segment joining $(0, 0)$, $(u + s, v + t)$ is of length $\sqrt{(u + s)^2 + (v + t)^2}$.
> The sum of the first two lengths is expected to be no shorter than the last. But this is expected: the three line segments are the three sides of the triangle with vertices $(0, 0)$, (u, v), $(u + s, v + t)$.
> Equality holds exactly when the three points (u, v), (s, t), $(u + s, v + t)$ are on the same 'half-line' with endpoint at the origin $(0, 0)$.



Proof. Postponed. (The 'classical method' is to first prove the *Cauchy-Schwarz Inequality*, of which Lemma (4) may be regarded as a special case, and then obtain the Triangle Inequality as a corollary.)