1. Consider the pair of statements (†), (‡) of the form below:

   (†) '*Let/suppose blih-blih-blih. Suppose blah-blah-blah. Then bleh-bleh-bleh.*'

   (‡) '*Let/suppose blih-blih-blih. Suppose bleh-bleh-bleh. Then blah-blah-blah.*'

   When want to state that both of (†), (‡) are to hold simultaneously, we may combine them into one statement of the form

   - '*Let/suppose blih-blih-blih. blah-blah-blah iff bleh-bleh-bleh.*'

   When one or both of '*blah-blah-blah*', '*bleh-bleh-bleh*' is very lengthy, we may write in this way:

   - '*Let/suppose blih-blih-blih. The following statements are logically equivalent:*

     (1) *Blah-blah-blah.*

     (2) *Bleh-bleh-bleh.*'

   The safest way for proving such a statement is to return to its original meaning: prove (†), (‡) separately.

2. **Statement (a).**

Suppose x, y are positive real numbers. Then $\dfrac{x+y}{2} = \sqrt{xy}$ *iff* $x = y$.

Proof of Statement (a).

Suppose x, y are positive real numbers. (Then $\sqrt{x}, \sqrt{y}, \sqrt{xy}, \sqrt{x} - \sqrt{y}$ are well-defined as real numbers.)

- ['$\Leftarrow$ -part'.] Suppose $x = y$. [We want to deduce $\frac{x+y}{2} = \sqrt{xy}$.]

  Then $\frac{x+y}{2} = \frac{2x}{2} = x$.

  Since x is positive, $\sqrt{x^2} = x$. Then $\sqrt{xy} = \sqrt{x^2} = x$.

  Hence $\frac{x+y}{2} = x = \sqrt{xy}$.

- ['$\Rightarrow$-part'.] Suppose $\dfrac{x+y}{2} = \sqrt{xy}$. [We want to deduce $x = y$.]

  Since x, y are positive, we have $(\sqrt{x})^2 = x$ and $(\sqrt{y})^2 = y$ and $\sqrt{x} \cdot \sqrt{y} = \sqrt{xy}$.

  Then $(\sqrt{x})^2 + (\sqrt{y})^2 = x + y = 2\sqrt{xy} = 2\sqrt{x} \cdot \sqrt{y}$.

  Therefore $(\sqrt{x} - \sqrt{y})^2 = (\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{x} \cdot \sqrt{y} = 0$.

  Hence $\sqrt{x} - \sqrt{y} = 0$.

  Now we have $\sqrt{x} = \sqrt{y}$. Therefore $x = (\sqrt{x})^2 = (\sqrt{y})^2 = y$. $\square$

3. **Statement (b).**

Let $\mathbf{x}$, $\mathbf{y}$ *be non-zero vectors in the real* n*-dimensional space. The following statements are logically equivalent:*

(1) *There exist some real numbers* κ, λ, *not both zero, such that* $\kappa\mathbf{x} + \lambda\mathbf{y} = \mathbf{0}$.

(2) $|\langle \mathbf{x}, \mathbf{y} \rangle| = \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.

Proof of Statement (b).

Let $\mathbf{x}$, $\mathbf{y}$ be vectors in the real n-dimensional space.

- [‘(1)⇒(2)’?]

  Suppose there exist some real numbers κ, λ, not both zero, such that $\kappa\mathbf{x}+\lambda\mathbf{y} = \mathbf{0}$.

  [We want to deduce $|\langle x, y \rangle| = \|x\| \cdot \|y\|$.]

  Without loss of generality, suppose $\lambda \neq 0$.

  Then $y = -\frac{\kappa}{\lambda} x$. Therefore

  $$|\langle x, y \rangle| = |\langle x, -\tfrac{\kappa}{\lambda} x \rangle| = |-\tfrac{\kappa}{\lambda}\langle x, x \rangle| = |-\tfrac{\kappa}{\lambda}| \cdot |\langle x, x \rangle|$$
  $$= |-\tfrac{\kappa}{\lambda}| \cdot \|x\|^2$$
  $$= \|x\| \cdot (|-\tfrac{\kappa}{\lambda}| \cdot \|x\|) = \|x\| \cdot \|-\tfrac{\kappa}{\lambda} x\| = \|x\| \cdot \|y\|.$$

- [‘(2)⇒(1)’?] ...

Statement (b).

Let $\mathbf{x}$, $\mathbf{y}$ *be non-zero vectors in the real* n*-dimensional space. The following statements are logically equivalent:*

(1) *There exist some real numbers* κ, λ, *not both zero, such that* $\kappa\mathbf{x} + \lambda\mathbf{y} = \mathbf{0}$.

(2) $|\langle \mathbf{x}, \mathbf{y} \rangle| = \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.

Proof of Statement (b).

Let $\mathbf{x}$, $\mathbf{y}$ be vectors in the real n-dimensional space.

- ['(1)⇒(2)'?] ...
- ['(2)⇒(1)'?]

  Suppose $|\langle \mathbf{x}, \mathbf{y} \rangle| = \|\mathbf{x}\| \cdot \|\mathbf{y}\|$. Then $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ or $\langle \mathbf{x}, \mathbf{y} \rangle = -\|\mathbf{x}\| \cdot \|\mathbf{y}\|$. [What do we want to deduce?]

(Case 1). Suppose $\langle x, y \rangle = \|x\| \cdot \|y\|$.

Define $\kappa = \|y\|$, $\lambda = -\|x\|$.

Since x, y are non-zero vectors, κ, λ are non-zero real numbers.

We have $\|\kappa x + \lambda y\|^2 = \langle \kappa x + \lambda y, \kappa x + \lambda y \rangle$

$= \kappa^2 \langle x, x \rangle + \kappa\lambda \langle x, y \rangle + \lambda\kappa \langle y, x \rangle + \lambda^2 \langle y, y \rangle = \ldots = 0$

[Your exercise.]

Then $\|\kappa x + \lambda y\| = 0$. Therefore $\kappa x + \lambda y = 0$.

(Case 2). Suppose $\langle x, y \rangle = -\|x\| \cdot \|y\|$. Then so-and-so. ← [Your exercise.]

4. Here are some other examples of such statements in school mathematics.

(α) *Let $\triangle ABC$ be a triangle.*
$\angle ACB$ is a right angle iff $AB^2 = AC^2 + BC^2$.

(β) *Let $\triangle ABC$ be a triangle.*
$\angle ACB$ is a right angle iff AB passes through the centre of the circumcircle of $\triangle ABC$.

(γ) *Let $f(z)$ be a polynomial with real/complex coefficients and indeterminate z, and c be a real/complex number.*
The polynomial $z - c$ is a factor of the polynomial $f(z)$ iff $f(c) = 0$.

(δ) *Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of complex numbers. The statements below are logically equivalent:*

(1) *$\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression. (There exists some complex number d such that for any $n \in \mathbb{N}$, $a_n = a_0 + nd$.)*

(2) *For any $n \in \mathbb{N}$, $a_{n+1} = \dfrac{a_n + a_{n+2}}{2}$.*

5. Many results in your *linear algebra* course are statements of this form. Here are some examples.

(α) *Let* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n \in \mathbb{R}^m$. *The statements below are logically equivalent:*

(1) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent.*

(2) *One of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of the others.*

(β) *Let* A *be an* $(n \times n)$*-square matrix with real entries. The statements below are logically equivalent:*

(1) A *is non-singular. (The zerovector is the only element of the null space of* A*.)*

(2) A *is row-equivalent to the identity matrix* I_n*.*

(3) A *is invertible.*

(4) *For any* $\mathbf{b} \in \mathbb{R}^n$*, the system* $A\mathbf{x} = \mathbf{b}$ *has a unique solution.*

(5) *The columns of* A *constitute a basis for* $\mathbb{R}^n$*.*

(6) $\det(A) \neq 0$.

Watch out how these results are proved in your *linear algebra* course.