1. We have been assuming these things tacitly since school days:

   (a) *The sum, the difference, and the product of any two (not necessarily distinct) integers are integers.*

   In symbols, this reads:

   Let $x, y \in \mathbb{Z}$. $x + y \in \mathbb{Z}$, $x - y \in \mathbb{Z}$, and $xy \in \mathbb{Z}$.

   (These 'operations' obey certain 'laws of arithmetic' which we have learnt and accepted since school days.)

   (b) *The sum and the product of any two (not necessarily distinct) positive integers are positive integers.*

   Moreover, every integer is either positive or negative or zero.

2. Definition.

Let $u, v \in \mathbb{Z}$.

This contains three pieces of information:
① The already present u, v 'generates' this k, whose exact value depends on the values of u, v.
② $k \in \mathbb{Z}$.
③ u, v, k are related by $u = kv$.

u is said to be **divisible** by v if there exists some $k \in \mathbb{Z}$ such that $u = kv$.

Remark. Before you start considering for a given pair of objects u, v whether it is true that u is divisible by v, you have to make sure that u, v are integers in the first place.

Examples.

- 6 is divisible by 2. Reason: $3 \in \mathbb{Z}$ and $6 = 3 \cdot 2$.
- 8 is divisible by -2. Reason: $-4 \in \mathbb{Z}$ and $8 = (-4) \cdot (-2)$

Further remark.

According to definition, 0 is divisible by 0. Reason: $1 \in \mathbb{Z}$ and $0 = 1 \cdot 0$.
However, no integer except 0 is divisible by 0.

Reason?
Claim: 'Let $u \in \mathbb{Z}$. Suppose u is divisible by 0. Then $u = 0$.'
Justification according to definition?

3. **Theorem (1) (Properties of divisibility).**

   The following statements hold:

   (a) *Let $x \in \mathbb{Z}$. x is divisible by x.*

   (b) *Let $x, y \in \mathbb{Z}$.*
   Suppose x is divisible by y and y is divisible by x.
   Then $|x| = |y|$.

   (c) *Let $x, y, z \in \mathbb{Z}$.*
   Suppose x is divisible by y and y is divisible by z.
   Then x is divisible by z.

   We are going to prove Statement (a) and Statement (c). The proof of Statement (b) is left as an exercise.

4. Proof of Statement (a) of Theorem (1).

Let $x \in \mathbb{Z}$. [What to prove? Un-wrap definition.]

[Want to prove: 'x is divisible by x.']

$x = 1 \cdot x$

Note that $1 \in \mathbb{Z}$.

[So there exists some $k \in \mathbb{Z}$, namely $k = 1$, such that $x = k \cdot x$.]

Hence x is divisible by x. ∎

Roughwork.

We actually want to prove:
'there exists some $k \in \mathbb{Z}$ such that $x = k \cdot x$.'

How to reach this objective?

- Name an appropriate k which simultaneously satisfies:
  $\begin{cases} (\text{✶}) & k \in \mathbb{Z} \\ (\text{✶✶}) & x = k \cdot x. \end{cases}$

Very formal proof of Statement (a) of Theorem (1).

I. Let $x \in \mathbb{Z}$. [Assumption.]

II. $x = 1 \cdot x$. [**I**, laws of arithmetic.]

III. $1 \in \mathbb{Z}$. [Property of the number 1.]

IV. $x = xq$ for some $q \in \mathbb{Z}$, namely $q = 1$. [**II**, **III**.]

V. x is divisible by x. [**IV**, definition of divisibility.]

5. Proof of Statement (c) of Theorem (1).

Let $x, y, z \in \mathbb{Z}$.

Suppose x is divisible by y and y is divisible by z. [What to deduce? What is the objective?]

[Want to deduce: 'x is divisible by z'.] $\longrightarrow$

Since x is divisible by y,
there exists some $g \in \mathbb{Z}$ such that $x = gy$.

Since y is divisible by z,
there exists some $h \in \mathbb{Z}$ such that $y = hz$.

Now
$$x = gy = g(hz) = (gh)z.$$
Since $g \in \mathbb{Z}$ and $h \in \mathbb{Z}$,
we have $gh \in \mathbb{Z}$.

[So there exists some $k \in \mathbb{Z}$,
namely $k = gh$, such that $x = kz$.]

Then x is divisible by z. ▫

Roughwork.

We actually want to deduce:
'there exists some $k \in \mathbb{Z}$ such that $x = kz$.'
How to reach this objective?

• Name an appropriate k which simultaneously satisfies:
$$\begin{cases} (\text{☆}) & k \in \mathbb{Z} \\ (\text{☆☆}) & x = k \cdot z. \end{cases}$$
But how to conceive such a 'k'?

• Look for candidate(s) for such a 'k' out of the information provided by the assumptions.

Further roughwork.

Now given: $g, h \in \mathbb{Z}$ and $x = gy$ and $y = hz$.
Can we relate x with z directly?
Yes: $x = gy = g(hz) = (gh)z.$

• So a good candidate for k is gh.

Very formal proof of Statement (c) of Theorem (1).

I. Let $x, y \in \mathbb{Z}$. [Assumption.]

II. Suppose x is divisible by y and y is divisible by z. [Assumption.]

III. x is divisible by y. [**II**.]

IV. There exists some $g \in \mathbb{Z}$ such that $x = gy$. [**III**, definition of divisibility.]

   IVi. $x = gy$. [**IV**.]

   IVii. $g \in \mathbb{Z}$. [**IV**.]

V. y is divisible by z. [**II**.]

VI. There exists some $h \in \mathbb{Z}$ such that $y = hz$. [**III**, definition of divisibility.]

   VIi. $y = hz$. [**VI**.]

   VIii. $h \in \mathbb{Z}$. [**VI**.]

VII. $x = gy$ and $y = hz$. [**IVi**, **VIi**.]

VIII. $x = ghz$. [**VII**.]

IX. $g \in \mathbb{Z}$ and $h \in \mathbb{Z}$. [**IVii**, **VIii**.]

X. $gh \in \mathbb{Z}$. [**I**, laws of arithmetic.]

XI. There exists some $k \in \mathbb{Z}$, namely, $k = gh$, such that $x = kz$. [**VIII**, **X**.]

XII. x is divisible by z. [**XI**, definition of divisibility.]

6. **Theorem (2) (Further properties of divisibility).**

Let $n \in \mathbb{Z}$. The following statements hold:

(a) *Let $x, y \in \mathbb{Z}$.*
Suppose x is divisible by n and y is divisible by n.
Then $x + y$ is divisible by n.

(b) *Let $x, y \in \mathbb{Z}$.*
Suppose x is divisible by n or y is divisible by n.
Then xy is divisible by n.

Proof. Exercise.