1. Tacitly assumed properties of the real number system since school-days:

   (a) i. *Let* $x, y \in \mathbb{R}$. $x + y \in \mathbb{R}$ *and* $x - y \in \mathbb{R}$ *and* $xy \in \mathbb{R}$.

   ii. *Let* $x, y \in \mathbb{R}$. *Suppose* $y \neq 0$. *Then* $x/y \in \mathbb{R}$.

   (b) i. *Let* $x \in \mathbb{R}$. *Exactly one of* '$x < 0$', '$x = 0$', '$x > 0$' *is true.*

   ii. *Let* $x, y \in \mathbb{R}$. *Suppose* $x > 0$ *and* $y > 0$. *Then* $x + y > 0$ *and* $xy > 0$ *and* $x/y > 0$.

   iii. *Let* $x, y \in \mathbb{R}$. *Suppose* $xy > 0$. *Then* $(x > 0$ *and* $y > 0)$ *or* $(x < 0$ *and* $y < 0)$.

   (c) *For each positive real number* x, *for each integer* $n \geq 2$, *there exists some positive real number* r *such that* $x = r^n$.

   We denote this r *by* $\sqrt[n]{x}$ *and call it the* n*-th real root of* x.

2. **Statement (A1).**

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

Proof of Statement (A1).

Ask: Assumptions in the statement? Conclusions?

Write down the assumptions. →

Let x, y be positive real numbers.

Suppose $x^2 > y^2$.

Then $x^2 - y^2 > 0$.

Note that $x^2 - y^2 = (x-y)(x+y)$.

Then $(x-y)(x+y) > 0$.

Therefore

$(x-y>0 \text{ and } x+y>0)$ or $(x-y<0 \text{ and } x+y<0)$.

Don't panic: which part of the assumptions is yet to be used? →

Since $\underline{\underline{x>0}}$ and $\underline{\underline{y>0}}$, we have $x+y>0$.

Then $x-y>0$ and $x+y>0$.

In particular $x-y>0$.

Therefore $x>y$. □

Roughwork.

Ask: What do we want to deduce?

Answer: '$x > y$'.

Ask: Any equivalent formulation which may be easier to manipulate and which may seem to link with the assumptions?

Answer: '$x - y > 0$'.

We observe:

① We can turn the assumption '$x^2 > y^2$' into '$x^2 - y^2 > 0$'.

② $\underbrace{x^2-y^2}_{\text{positive by assumption.}} = \underbrace{(x-y)}_{???}\underbrace{(x+y)}_{\text{positive ???}}$.

Statement (A1).

Let x, y be positive real numbers. Suppose $x^2 > y^2$. Then $x > y$.

Very formal proof of Statement (A1).

I. Let x, y be positive real numbers. [Assumption.]

II. Suppose $x^2 > y^2$. [Assumption.]

III. $x^2 - y^2 > 0$. [**II**.]

IV. $x^2 - y^2 = (x - y)(x + y)$. [Properties of the reals.]

V. $(x - y)(x + y) > 0$. [**III**, **IV**.]

VI $(x - y > 0$ and $x + y > 0)$ or $(x - y < 0$ and $x + y < 0)$. [**V**, properties of the reals.]

VII. $x + y > 0$ [**I**.]

VIII. $x - y > 0$. [**VI**, **VII**.]

IX. $x > y$. [**VIII**.]

3. **Statement (A2).**

Let x, y be positive real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

Proof of Statement (A2).

Let x, y be positive real numbers. Suppose $x^2 \geq y^2$.
Then $x^2 - y^2 \geq 0$.
Note that $x^2 - y^2 = (x - y)(x + y)$.
Then $(x - y)(x + y) \geq 0$.

Since $x > 0$ and $y > 0$, we have $x + y > 0$. Therefore $\dfrac{1}{x+y} > 0$ also.

Then $x - y = [(x - y)(x + y)] \cdot \dfrac{1}{x+y} \geq 0$.

Therefore $x \geq y$.

4. **Statement (B).**

Suppose x, y are positive real numbers. Then $\frac{x+y}{2} \geq \sqrt{xy}$.

Proof of Statement (B). [Assumptions? Conclusion?]

[Assumption] → Suppose x, y are positive real numbers.

Here we make sure that everything which appears in the calculation below makes sense.

→ Then $\sqrt{x}$, $\sqrt{y}$ are well-defined as real numbers.

→ Also, $\sqrt{x}-\sqrt{y}$ is well-defined as a real number.

→ Since x, y are positive, xy is also positive.

→ Moreover, $\sqrt{xy}$ is well-defined as a real number, and $\sqrt{xy} = \sqrt{x}\cdot\sqrt{y}$.

→ Since x, y are positive, $x = (\sqrt{x})^2$ and $y = (\sqrt{y})^2$.

Therefore

$$x+y-2\sqrt{xy} = (\sqrt{x})^2 + (\sqrt{y})^2 - 2\sqrt{x}\cdot\sqrt{y}$$
$$= (\sqrt{x}-\sqrt{y})^2$$
$$\geq 0.$$

Hence $\frac{x+y}{2} \geq \sqrt{xy}$. □

Ask: How to reach '$\frac{x+y}{2} \geq \sqrt{xy}$'?

Clueless? So ask:

Is there some equivalent formulation of '$\frac{x+y}{2} \geq \sqrt{xy}$' which is more suggestive, linking what we know or have learnt?

Or is there some consequence of '$\frac{x+y}{2} \geq \sqrt{xy}$' which is more suggestive?

Ask: Assuming $\frac{x+y}{2} \geq \sqrt{xy}$ holds, what happens?

Answer. $\frac{x+y}{2} \geq \sqrt{xy}$.

Then $x - 2\sqrt{xy} + y \geq 0$.

(Allowed?) → $(\sqrt{x})^2 - 2\sqrt{x}\cdot\sqrt{y} + (\sqrt{y})^2 \geq 0$.

$(\sqrt{x}-\sqrt{y})^2 \geq 0$. ← Suggestive?

Now ask: Can this process be 'reversed'?

Statement (B).

Suppose x, y are positive real numbers. Then $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Very formal proof of Statement (B).

I. Suppose x, y are positive real numbers. [Assumption.]

II. $\sqrt{x}, \sqrt{y}$ are well-defined as real numbers. [**I**.]

III. $\sqrt{x} - \sqrt{y}$ is well-defined as a real number. [**II**.]

IV. xy is a positive real number. [**I**, properties of the reals.]

V. $\sqrt{xy}$ is well-defined as a real number. [**IV**.]

VI. $\sqrt{x}\sqrt{y} = \sqrt{xy}$. [**II**, **V**, properties of the reals.]

VII. $(\sqrt{x})^2 = x$. [**I**, **II**.]

VIII. $(\sqrt{y})^2 = y$. [**I**, **II**.]

IX. $(\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y$. [**VI**, **VII**, **VIII**.]

X.$(\sqrt{x} - \sqrt{y})^2 \geq 0$. [**III**, properties of the reals.]

XI. $x - 2\sqrt{xy} + y \geq 0$. [**IX**, **X**.]

XII. $\dfrac{x+y}{2} \geq \sqrt{xy}$. [**XI**.]

5. **Statement (C).**

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$. Then $x^2 + xy + y^2 > 0$.

Proof of Statement (C).

Let $x, y \in \mathbb{R}$. Suppose $x \neq 0$ or $y \neq 0$.

(Case 1). Suppose $x \neq 0$.

Then
$$x^2+xy+y^2 = \frac{3x^2}{4} + \left(\frac{x}{2}+y\right)^2 > 0 + 0 = 0.$$

(Case 2). Suppose $y \neq 0$.

Then
$$x^2+xy+y^2 = \frac{3y^2}{4} + \left(\frac{y}{2}+x\right)^2 > 0 + 0 = 0.$$

Hence, in any case, $x^2+xy+y^2>0$. □

Smart argument.

Note that $x^2+xy+y^2 = \frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}(x+y)^2$. So ???

Ask: How to reach '$x^2+xy+y^2>0$' from '$x \neq 0$'?

Answer: Observe that
$$x^2+xy+y^2$$
is a quadratic expression.

This suggests something we have learnt:

Completing the square.

Ask: Is the equality below possible?

$$x^2+xy+y^2 = \#_1 \cdot x^2 + \#_2 \cdot \underbrace{(\cdots\cdots)}_{\text{absorbing everything involving } y\text{?}}{}^2$$

non-negative numbers?

Answer: Yes:

$$x^2+xy+y^2 = \frac{3}{4}x^2 + 1\cdot\left(\frac{x}{2}+y\right)^2$$

And this is positive because $x \neq 0$.

Reminder: What if $x=0$?

6. **Statement (A').**

Let x, y be non-negative real numbers. Suppose $x^2 \geq y^2$. Then $x \geq y$.

Proof of Statement (A').

Let x, y be non-negative real numbers.

Suppose $x^2 \geq y^2$.

Then $x^2 - y^2 \geq 0$.

Note that $x^2 - y^2 = (x-y)(x+y)$.

Then $(x-y)(x+y) \geq 0$.

Therefore

why? So we realize something we have tacitly assumed in school maths.

$(x-y \geq 0$ and $x+y \geq 0)$ or $(x-y \leq 0$ and $x+y \leq 0)$.

It is tempting to immediately conclude from '$x+y \geq 0$' that '$x-y \geq 0$'. But there is a problem. ...

Since $x \geq 0$ and $y \geq 0$, we have $x+y \geq 0$.

Then $x+y > 0$ or $x+y = 0$.

(Case 1). Suppose $x+y>0$.
Since $(x-y)(x+y) \geq 0$, we have $x-y \geq 0$. Then $x \geq y$.

(Case 2). Suppose $x+y=0$.
Since $x \geq 0$ and $y \geq 0$, we have $x=y=0$. Then $x \geq y$.

Therefore, in any case, we have $x \geq y$. □

Very formal proof of Statement (A').

I. Let x, y be non-negative real numbers. [Assumption.]

II. Suppose $x^2 \geq y^2$. [Assumption.]

III. $x^2 - y^2 \geq 0$. [**II**.]

IV. $x^2 - y^2 = (x-y)(x+y)$. [Properties of the reals.]

V. $(x-y)(x+y) \geq 0$. [**III, IV**.]

VI $(x - y \geq 0$ and $x + y \geq 0)$ or $(x - y \leq 0$ and $x + y \leq 0)$. [**V**, properties of the reals.]

VII. $x + y \geq 0$. [**I**.]

VIII. $x+y > 0$ or $x+y = 0$. [**VII**.]

IX.

- **IXi**. Suppose $x + y > 0$. [One of the possibilities in **VIII**.]
- **IXii**. $x - y \geq 0$. [**VI, IXi**.]
- **IXiii**. $x \geq y$. [**IXii**.]

X.

- **Xi**. Suppose $x + y = 0$. [One of the possibilities in **VIII**.]
- **Xii**. $x = y = 0$. [**I, Xi**.]
- **Xiii**. $x \geq y$. [**Xii**.]

XI. $x \geq y$. [**VIII, IX, X**.]

7. **Statement (D). (Bernoulli's Inequality.)**

Let $m \in \mathbb{N}\backslash\{0,1\}$ *and* $\beta \in \mathbb{R}$. *Suppose* $\beta > 0$ *or* $-1 < \beta < 0$. *Then* $(1+\beta)^m > 1 + m\beta$.

Proof of Statement (D).

Let $m \in \mathbb{N}\backslash\{0,1\}$ and $\beta \in \mathbb{R}$.

Suppose $\beta > 0$ (or) $-1 < \beta < 0$.

[Want to deduce: $(1+\beta)^m > 1+m\beta$.]

Roughwork.

Ask: How to arrive at '$(1+\beta)^m > 1+m\beta$'? Any equivalent formulation which links up with what we learnt?

Answer: '$(1+\beta)^m - 1 > m\beta$'.

Recall from school maths:

$$s^n - t^n = (s-t)(s^{n-1} + s^{n-2}t + s^{n-3}t^2 + \dots + s^2t^{n-3} + st^{n-2} + t^{n-1})$$

So?

Note that

$$\begin{aligned}(1+\beta)^m - 1 &= (1+\beta)^m - 1^m \\ &= [(1+\beta)-1][(1+\beta)^{m-1} + (1+\beta)^{m-2} + \dots + (1+\beta) + 1] \\ &= \beta \cdot [(1+\beta)^{m-1} + (1+\beta)^{m-2} + \dots + (1+\beta) + 1].\end{aligned}$$

(Case 1). Suppose $\beta > 0$. Then, since $\beta > 0$ and $1+\beta > 1$,

$$(1+\beta)^m - 1 = \beta \cdot \underbrace{[(1+\beta)^{m-1} + (1+\beta)^{m-2} + \dots + (1+\beta) + 1]}_{m \text{ terms}} > \beta \cdot \underbrace{(1+1+\dots+1+1)}_{m \text{ copies}} = m\beta.$$

(Case 2). Suppose $-1 < \beta < 0$. Then, since $-\beta > 0$ and $0 < 1+\beta < 1$,

$$1 - (1+\beta)^m = (-\beta) \cdot [(1+\beta)^{m-1} + (1+\beta)^{m-2} + \dots + (1+\beta) + 1] < (-\beta) \cdot \underbrace{(1+1+\dots+1+1)}_{m \text{ copies}} = -m\beta.$$

Therefore, in any case, $(1+\beta)^m > 1 + m\beta$. □

Remark. Below is a more general version of **Bernoulli's Inequality**:

Let μ be a rational number, and β be a real number.

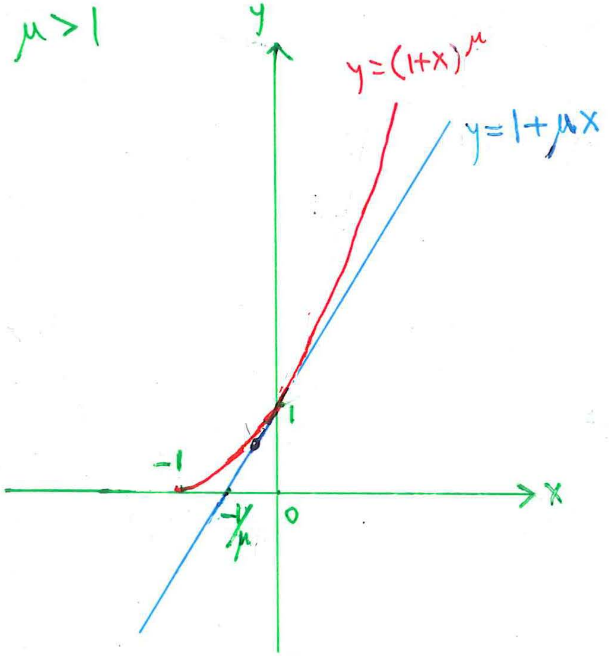
Suppose $\mu \neq 0$ and $\mu \neq 1$, and $\beta > -1$.

The statements below hold:

(1) *Suppose $\mu < 0$ or $\mu > 1$. Then $(1+\beta)^\mu \geq 1 + \mu\beta$.*

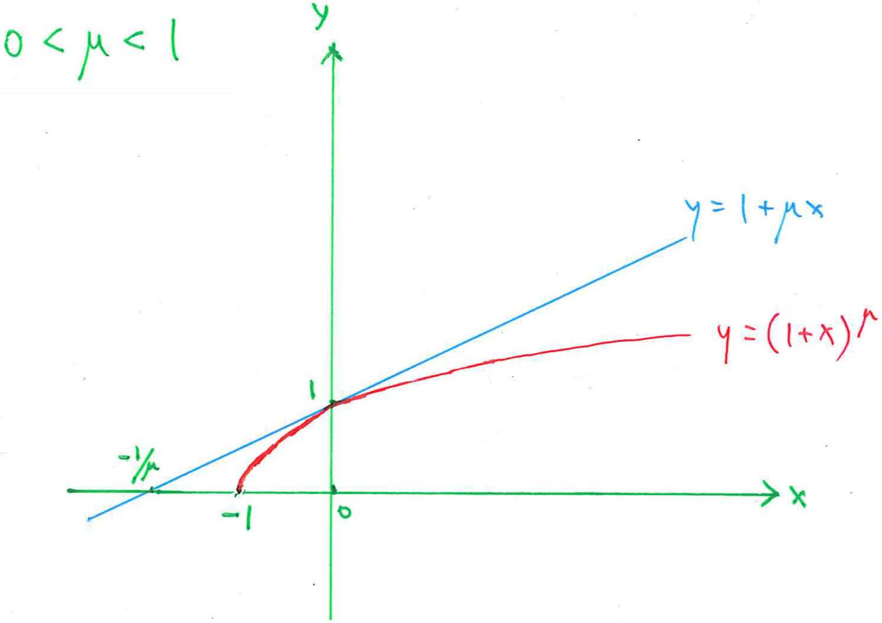
(2) *Suppose $0 < \mu < 1$. Then $(1+\beta)^\mu \leq 1 + \mu\beta$.*

(3) *In each of (1), (2), equality holds iff $\beta = 0$.*



$(1+x)^\mu > 1+\mu x$

whenever $-1<x<0$ or $x>0$.



$(1+x)^\mu < 1+\mu x$

whenever $-1<x<0$ or $x>0$.

8. We need expand the list of 'rules as regards inequalities' which we are tacitly assuming since school-days!

(1) *Let* $x, y \in \mathbb{R}$. $y - x > 0$ *iff* $x < y$.

(1*) *Let* $x, y \in \mathbb{R}$. $y - x \geq 0$ *iff* $x \leq y$.

(2) *Let* $x, y, z \in \mathbb{R}$. *If* $x < y$ *and* $y < z$ *then* $x < z$.

(2*) *Let* $x, y, z \in \mathbb{R}$. *The statements below hold:*

- (2*a) $x \leq x$.
- (2*b) *If* ($x \leq y$ *and* $y \leq x$) *then* $x = y$.
- (2*c) *If* ($x \leq y$ *and* $y \leq z$) *then* $x \leq z$.

(3) *Let* $x \in \mathbb{R}$. *Exactly one of* '$x < 0$', '$x = 0$', '$x > 0$' *is true.*

(4) *Let* $x, y \in \mathbb{R}$. *Suppose* $x < y$. *Then the statements below hold:*

- (4a) *For any* $u \in \mathbb{R}$, $x + u < y + u$ *and* $x - u < y - u$.
- (4b) *For any* $u \in \mathbb{R}$, *if* $u > 0$ *then* $xu < yu$ *and* $x/u < y/u$.
- (4c) *For any* $u \in \mathbb{R}$, *if* $u < 0$ *then* $xu > yu$ *and* $x/u > y/u$.

(4*) *Let* $x, y \in \mathbb{R}$. *Suppose* $x \leq y$. *Then the statements below hold:*

- (4*a) *For any* $u \in \mathbb{R}$, $x + u \leq y + u$ *and* $x - u \leq y - u$.
- (4*b) *For any* $u \in \mathbb{R}$, *if* $u > 0$ *then* $xu \leq yu$ *and* $x/u \leq y/u$.
- (4*c) *For any* $u \in \mathbb{R}$, *if* $u < 0$ *then* $xu \geq yu$ *and* $x/u \geq y/u$.

... More rules:

(5) *Let* $x, y, u, v \in \mathbb{R}$. *Suppose* $x < y$ *and* $u < v$. *The statements below hold:*

(5*a*) $x + u < y + v$.

(5*b*) *Further suppose* $x > 0$, $y > 0$, $u > 0$ *and* $v > 0$. *Then* $xu < yv$.

(5*) *Let* $x, y, u, v \in \mathbb{R}$. *Suppose* $x \leq y$ *and* $u \leq v$.

(5**a*) $x + u \leq y + v$.

(5**b*) *Further suppose* $x \geq 0$, $y \geq 0$, $u \geq 0$ *and* $v \geq 0$. *Then* $xu \leq yv$.

(6) *Let* $x, y \in \mathbb{R}$. *The statements below hold:*

(6*a*) *Suppose* $xy > 0$. *Then* ($x > 0$ *and* $y > 0$) *or* ($x < 0$ *and* $y < 0$).

(6*b*) *Suppose* $xy < 0$. *Then* ($x > 0$ *and* $y < 0$) *or* ($x < 0$ *and* $y > 0$).

(6*) *Let* $x, y \in \mathbb{R}$. *The statements below hold:*

(6**a*) *Suppose* $xy \geq 0$. *Then* ($x \geq 0$ *and* $y \geq 0$) *or* ($x \leq 0$ *and* $y \leq 0$).

(6**b*) *Suppose* $xy \leq 0$. *Then* ($x \geq 0$ *and* $y \leq 0$) *or* ($x \leq 0$ *and* $y \geq 0$).

(7) *Let* $x \in \mathbb{R}$. *Suppose* $x \neq 0$. *Then* $x^2 > 0$.

(7*) *Let* $x \in \mathbb{R}$. $x^2 \geq 0$.