1. **Mathematical statements.**

   A **mathematical statement** is a sentence with mathematical content, for which it is meaningful to say it is true or it is false.

   These are some special features in mathematical statements, in contrast to sentences in 'everyday language':

   (a) Words or phrases with specific mathematical content have to be defined carefully.

   (b) Some words which appear in everyday language may have very specific meaning.

   (c) Very often ordering of words matters in the meaning.

   There is no room for inaccuracies. There is no room for ambiguities.

2. **Various types of mathematical statements.**

   Several types of statements deserve special attention when we encounter them:

   - Some mathematical statements are called **definitions**. Such a statement informs the reader of the mathematical meaning of a specific mathematical term/phrase. We do not have to justify whether a definition is true; they are by default regarded as true statements. (There are, however, good or bad definitions. The good ones are those which are 'mathematically useful'; this involves mathematical and philosophical judgement.)
   - Mathematical statements whose content are regarded to be important are called **conjectures** until they are proved or dis-proved on the basis of what is regarded to be more fundamental. When such a statement is proved, it is called a **theorem** (or, depending on the context, called a proposition or called a lemma).
   - However, there are a few mathematical statements which are agreed by many mathematicians to be so fundamental that their validity is assumed without proof. Such statements are called **axioms**.

3. **Examples of mathematical statements.**

   (a) $1 + 1 = 2$.

   (b) $1 + 2 = 4$.

   Remark. (a), (b) are both mathematical statements. One is true and the other is false.

   (c) *$\sqrt{2}$ is an irrational number.*

   Remark. The phrase 'irrational number' should be defined carefully.

   (d) **Pythagoras' Theorem**.

   Let ΔABC be a triangle. Suppose $\angle ACB$ is a right angle. Then $AB^2 = AC^2 + BC^2$.

   Remark. This is not a sentence. However, we may condense it into

   '*if (ΔABC is a triangle and $\angle ACB$ is a right angle) then $AB^2 = AC^2 + BC^2$*'.

   For this reason, we still regard these three inter-related sentences as a mathematical statement.

   The words between 'if' and 'then' form the **assumptions** of the statement. The words after 'then' form the **conclusion** of the statement.

   In principle we can 'condense' mathematical statements into a single sentence, most probably in the form 'if ... then ...', announcing the assumptions and conclusions, but we don't want to do so because it is too clumsy. We tend to arrange a lengthy mathematical statement into a number of carefully worded inter-related sentences.

   (e) **Converse of Pythagoras' Theorem**.

   Let ΔABC be a triangle. Suppose $AB^2 = AC^2 + BC^2$. Then $\angle ACB$ is a right angle.

   Remark. The mathematical contents of (d), (e) are different. We have interchanged the position of the words between 'suppose', 'then' and the position of the words after 'then'.

   (f) **Thales' Theorem**.

   Let Γ be a circle, and A, B, C be three distinct points on the circumference of Γ. Suppose AB passes through the centre of Γ. Then $\angle ACB$ is a right angle.

   Remark. Converse of Thales' Theorem reads:

'*Let Γ be a circle, and A, B, C be three distinct points on the circumference of Γ. Suppose $\angle ACB$ is a right angle. Then AB passes through the centre of Γ.*'

Further remark. When a mathematical statement and its converse are both correct mathematical results, we tend to combine them in the form 'blah-blah-blah iff bleh-bleh-bleh'. **Examples**:

- *Let ΔABC be a triangle. $\angle ACB$ is a right angle iff $AB^2 = AC^2 + BC^2$.*
- *Let Γ be a circle, and A, B, C be three distinct points on the circumference of Γ. AB passes through the centre of Γ iff $\angle ACB$ is a right angle.*

(g) **Fermat's Horizontal Tangent Theorem**.

Let I be an open interval, c be a point in I, and $f : I \longrightarrow \mathbb{R}$ be a function which is differentiable at c. Suppose f attains a relative extremum at c. Then $f'(c) = 0$.

Remark. The converse of Fermat's Horizontal Tangent Theorem is false. (Why? Start by writing down the statement in question.)

4. **Predicates**.

A **predicate with variables** $x, y, z, \cdots$ is a statement 'modulo' the ambiguity of possibly one or several variables $x, y, z, \cdots$. In general, it may fail to be a statement. However, provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

The key examples of predicates in school mathematics are:

- Equations with one unknown (or more).
- Inequalities with one unknown (or more).

We can also construct a statement out of a predicate by the use of quantifiers which eliminate the ambiguity due to the presence of the variables.

5. **Equations and inequalities as predicates.**

(a) Every **equation** with one unknown (or more) is a predicate in which the variables are the unknowns of the equation.
Examples.

(i) $x^2 - 1 = 0$.
(ii) $x^2 + 1 = 0$.
(iii) $x + 2y + 3 = 0$.
(iv) $x^2 + y^2 = 1$.
(v) $x^2 - y^2 = 1$.
(vi) $x + y + z = 1$.
(vii) $x^2 + y^2 + z^2 = 1$.
(viii) $x^2 + y^2 = z^2$.

(b) Every **inequality** with one unknown (or more) is a predicate in which the variables are the unknowns of the inequality.
Examples.

(i) $x^2 - 1 \geq 0$.
(ii) $x^2 + 1 > 0$.
(iii) $x + 2y + 3 \leq 0$.
(iv) $x^2 + y^2 < 1$.
(v) $x^2 - y^2 < 1$.
(vi) $x + y + z \geq 1$.
(vii) $x^2 + y^2 + z^2 \leq 1$.
(viii) $x^2 + y^2 \leq z^2$.

6. **What is 'solving an equation/inequality'?**

To **solve** an equation/inequality with unknowns $x, y, z, \cdots$ amongst so-and-so is to specify, for that equation/inequality regarded as a predicate with variables $x, y, z, \cdots$, all the 'concrete objects' amongst so-and-so which, upon 'substitution into the variables' of the predicate, turn the predicate into a true statement. Each such 'concrete object' which turn the predicate into a true statement is called a **solution** for that equation/inequality.

In practice, this is what we usually do:

- First perform some manipulation, starting from the equation/inequality concerned, in order to find all possible candidates for $x, y, z, \cdots$.
- Then substitute these candidates for $x, y, z, \cdots$ into the predicate (which is the equation/inequality concerned) to see whether we obtain a true statement.

For more detailed discussion, refer to the Handout *Solving equations and inequalities.*

7. **Further Examples on mathematical statements and predicates.**

(a) *For any* $n \in \mathbb{Z}$, $n^2 + n$ *is an even integer.*

Remark. At school we were more used to write this statement as:

'*Let* $n \in \mathbb{Z}$. $n^2 + n$ *is an even integer.*'

(b) **Bernoulli's Inequality.**

For any $m \in \mathbb{N} \backslash \{0, 1\}$, *for any* $\beta \in (-1, +\infty) \backslash \{0\}$, $(1 + \beta)^m > 1 + m\beta$.

Remark. At school we were more used to write this statement as:

'*Let* $m \in \mathbb{N} \backslash \{0, 1\}$. *Let* $\beta \in (-1, +\infty) \backslash \{0\}$. $(1 + \beta)^m > 1 + m\beta$.'

(c) **Mean-Value Theorem.**

Let $a, b \in \mathbb{R}$. *Suppose* $a < b$. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a function. Suppose* f *is continuous on* $[a, b]$ *and* f *is differentiable on* (a, b). *Then there exists some* $c \in (a, b)$ *such that* $f(b) - f(a) = (b - a)f'(c)$.

Remark. What is the assumption? What is the conclusion? What is the meaning of 'there exists'?

(d) **Definition of differentiability.**

Let I *be an open interval, and* $c \in I$. *Let* $f : I \longrightarrow \mathbb{R}$ *be a function.* f *is said to be differentiable at* c *if the limit* $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ *exists in* $\mathbb{R}$.

Remarks.

(A) The words '*said to be*', '*called*' in such a statement are 'signatures of definition'. They remind us that we are reading a mathematical definition.

(B) In this example, the words '*the limit ... exists in* $\mathbb{R}$' are the 'defining condition' in the definition. It is connected with the rest of the statement by the word '*if*'. It conveys the mathematical meaning of the phrase 'being differentiable at a point'.

(C) Because the 'definining condition' conveys the mathematical meaning of the term/phrase under question, we have to pay extra attention when reading/writing such words. You are not going to tamper with it unless you are absolutely sure that you have not changed its mathematical content.

(e) **Definition of divisibility.**

Let $a, b \in \mathbb{Z}$. a *is said to be divisible by* b *if there exists some* $k \in \mathbb{Z}$ *such that* $a = kb$.

Remark. We may also write 'there exists some $k \in \mathbb{Z}$ such that $a = kb$' as '$a = kb$ for some $k \in \mathbb{Z}$'. It is incorrect to write '$a = kb, k \in \mathbb{Z}$'.

(f) **Definition of even-ness and odd-ness for integers.**

Let $n \in \mathbb{Z}$.

* n *is said to be even if* n *is divisible by* 2.
* n *is said to be odd if* n *is not divisible by* 2.

Remark. So one definition is built upon another.

(g) **Definition of prime numbers.**

Let $p \in \mathbb{Z}$. *Suppose* $|p| \geq 2$. p *is said to be a prime number if the following condition holds:*

- *for any* $k \in \mathbb{Z}$, *if* p *is divisible by* k *then* ($k = 1$ *or* $k = -1$ *or* $k = p$ *or* $k = -p$).

Remark. The word '*if*' appears twice in this definition.

* The word '*if*' in '*if the following condition holds*' connects the 'definining condition' which explains the phrase 'being a prime number' to the rest of the statement of definition.
* The word '*if*' in '*if* p *is divisible by* k' is an integral part of the 'definining condition'.

They have different roles and meanings, and neither can be omitted.

(h) **Division Algorithm (for natural numbers).**

Let $a, b \in \mathbb{N}$. *Suppose* $a \neq 0$. *Then there exist some unique* $q, r \in \mathbb{N}$ *such that* $b = qa + r$ *and* $0 \leq r < a$.

Remarks.

(A) Such a statement is an example of existence-and-uniqueness statements. It is condensed from the following statement:

'Let $a, b \in \mathbb{N}$. Suppose $a \neq 0$. Then there exist some $q, r \in \mathbb{N}$ such that $b = qa + r$ and $0 \leq r < a$. Moreover, for any $s, s', t, t' \in \mathbb{N}$, if ($b = sa + t$ and $0 \leq s' < a$) and ($b = s'a + t'$ and $0 \leq s < a$) then ($s = s'$ and $t = t'$).'

(B) Why are we bothered with carefully formulating such a statement? The reason is that we want to give a justification for the statement concerned that satisfies mathematicians.
Why is it necessary to justify it? We know that when we 'divide' 13 by 3, we obtain 'the' 'quotient' 4 and 'the' 'remainder' 1. But how about 'dividing', say, $(1234567^{7654321})^{13579} + 97531^{1234567}$, by, 123456789? Running computer programme? But what if we don't have enough computing capacity? From the mathematicians' point of view, it is not enough to simply say that '*we can do it when the number concerned is small*', or give a demonstration on how it can be done '*when the number concerned is small*'.
Mathematicians want 'mathematical proofs'. In the example of the Division Algorithm, the mathematicians want to see why, for every value of a, b, what the statement says for those a, b describes is true. (There is a subtle difference in mentality between the mathematicians and the engineers.)

(i) **Definition of arithmetic progression.**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers. $\{a_n\}_{n=0}^{\infty}$ is said to be an arithmetic progression if there exists some number d such that for any $n \in \mathbb{N}$, $a_{n+1} - a_n = d$. The number d is called the common difference of this arithmetic progression.

Remark. This is nothing but a more formal way to state what you have learnt about arithmetic progression in school mathematics. Though it looks clumsy, it is a necessary evil: only with a carefully fomulated definition in place can we start to proving what we have always believed to be true but never justified as so.

(j) **Definition of geometric progression.**

Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero numbers. $\{b_n\}_{n=0}^{\infty}$ is said to be a geometric progression if there exists some non-zero number r such that for any $n \in \mathbb{N}$, $\dfrac{b_{n+1}}{b_n} = r$. The number r is called the common ratio of this geometric progression.