1. **Mathematical statements**.

A **mathematical statement** is a sentence with mathematical content, for which it is meaningful to say it is true or it is false.

Examples : ① 'divisible', 'even'.
② 'rational', 'irrational'.
③ 'differentiable'.

Some special features in mathematical statements :

(a) Words or phrases with specific mathematical content have to be defined carefully.

(b) Some words which appear in everyday language may have very specific meaning.

Examples : ① 'if ... then ...'
② 'or'
③ 'and'
④ 'every', 'each'.

(c) Very often ordering of words matters in the meaning.

Examples :
① 'For any ..., if ... then ...' versus 'If ... for any ... then ...'.
② 'For any ... there exits ...' versus 'there exits ... for any ...'.

No room for inaccuracies.

No room for ambiguities.

2. **Various types of mathematical statements**.

These which deserve attention:

- Definitions. ← Good/useful or bad/useless only; by default true.
  Good exactly when mathematically useful.
- Conjectures. Theorems. Propositions. Lemma. ← We look for proofs or dis-proofs for such statements.
- Axioms.
  ↳ Accepted to be true, without proof.
  Mathematically useful when accepted to be true.

3. **Examples of mathematical statements.**

(a) $1 + 1 = 2$.

(b) $1 + 2 = 4$.

Both are statements.

(a) is true. (b) is false. But both are statements.

(c) $\sqrt{2}$ is an *irrational* number. This word needs be defined carefully.

(d) **Pythagoras' Theorem**.

Let ΔABC be a triangle.

Suppose $\angle ACB$ is a right angle. ← Assumption

Then $AB^2 = AC^2 + BC^2$. ← Conclusion

Words indicating logical structure

More formal formulation:

For any triangle ΔABC, if $\angle ACB$ is a right angle then $AB^2 = AC^2 + BC^2$.

← Assumption ← Conclusion

(e) **Converse of Pythagoras' Theorem**.

Let ΔABC be a triangle.

Suppose $AB^2 = AC^2 + BC^2$. ← Assumption

Then $\angle ACB$ is a right angle. ← Conclusion

Words indicating logical structure

More formal formulation:

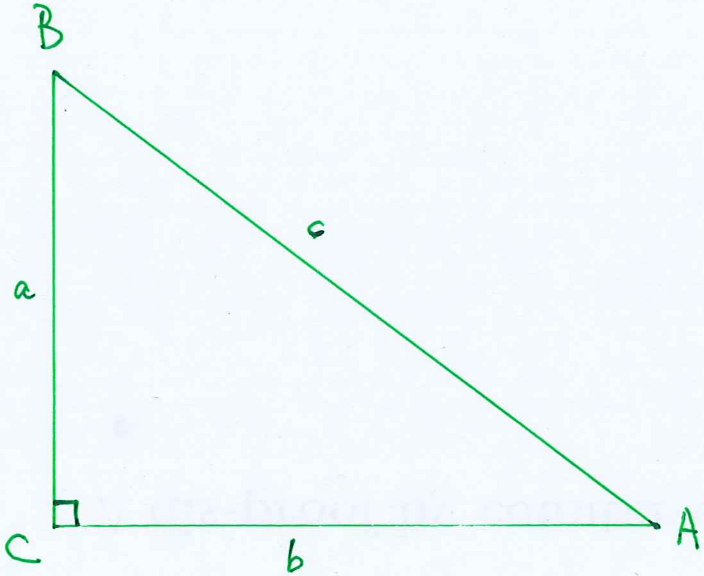
For any triangle ΔABC, if $AB^2 = AC^2 + BC^2$ then $\angle ACB$ is a right angle.

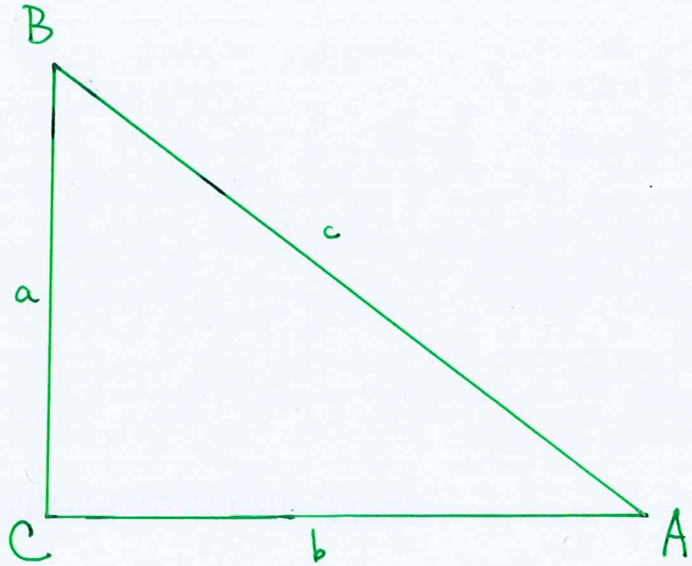
← Assumption ← Conclusion

Pythagoras' Theorem

Assumption:
$\angle ACB$ is a right angle.

Conclusion:
$a^2 + b^2 = c^2$.





Conclusion:
$\angle ACB$ is a right angle.

Assumption:
$a^2 + b^2 = c^2$

Converse to Pythagoras' Theorem

(f) **Thales' Theorem**.

Let Γ be a circle, and A, B, C be three distinct points on the circumference of Γ.
Suppose AB passes through the centre of Γ.
Then $\angle ACB$ is a right angle.

Assumption ?

Conclusion ?

Converse of Thales' Theorem :

Let Γ be a circle, and A, B, C be three distinct points on the circumference of Γ.

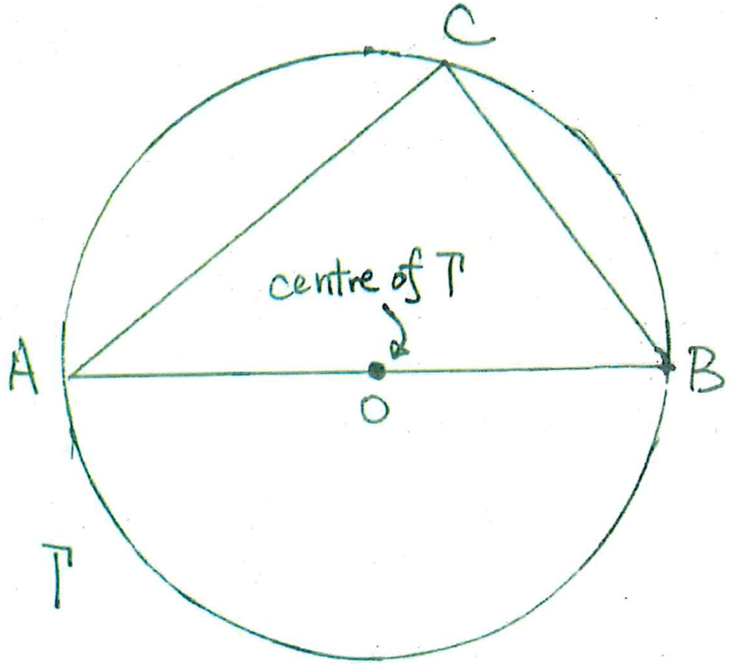
Suppose $\angle ACB$ is a right angle.

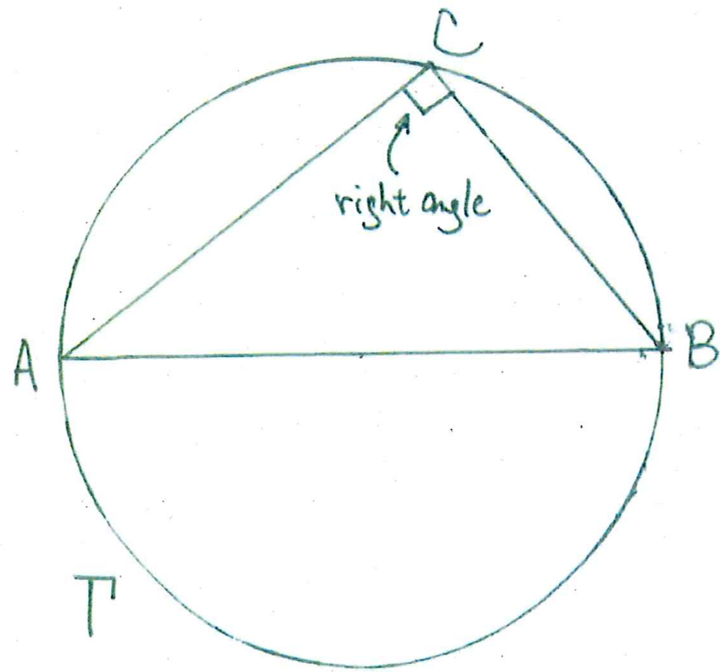
Then AB passes through the centre of Γ.

Thales' Theorem.

Assumption:
AB passes through the centre of Γ.

Conclusion:
$\angle ACB$ is a right angle.





Conclusion:
AB passes through the centre of Γ.

Assumption:
$\angle ACB$ is a right angle.

Converse of Thales' Theorem.

(g) **Fermat's Horizontal Tangent Theorem.**

Let I be an open interval, c be a point in I, and $f : I \longrightarrow \mathbb{R}$ be a function which is differentiable at c. Suppose f attains a relative extremum at c. Then $f'(c) = 0$.

Assumption?

Conclusion?

Converse of Fermat's Horizontal Tangent theorem:

Let I be an open interval, c be a point in I, and $f : I \to \mathbb{R}$ be a function which is differentiable at c.

Suppose $f'(c) = 0$.

Then f attains a relative extremum at c.

This is a false statement.

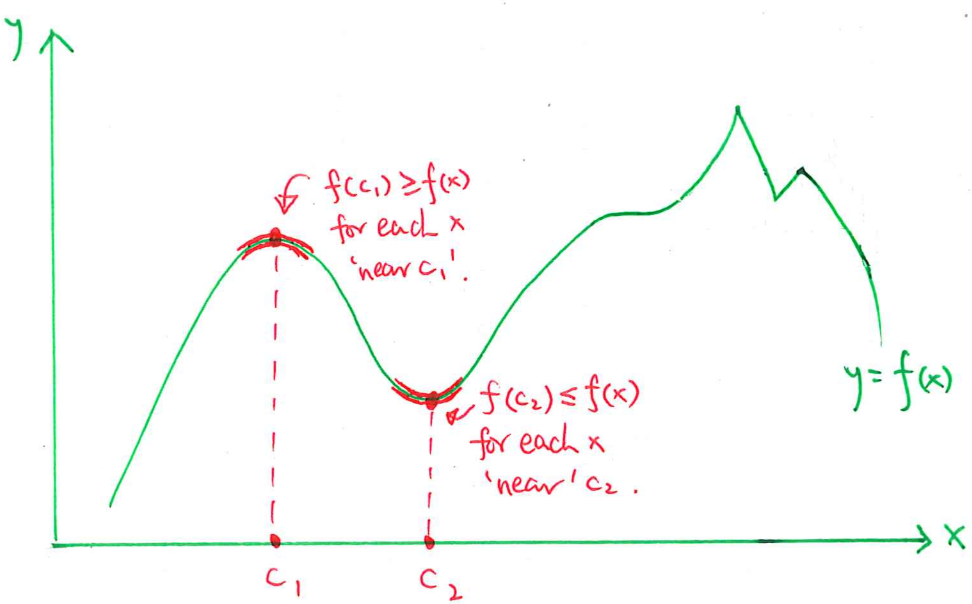
Fermat's Horizontal Tangent Theorem

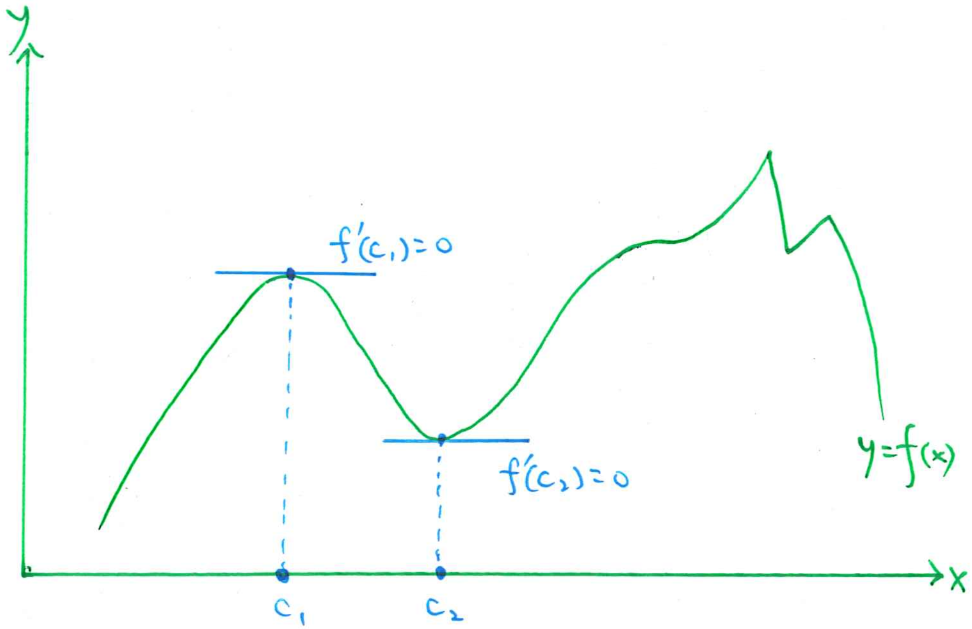
Assumption:

f attains a relative extremum at c (at which f is differentiable).

Conclusion:

$f'(c) = 0$.





4. **Predicates.**

A **predicate with variables** $x, y, z, \cdots$ is a statement 'modulo' the ambiguity of possibly one or several variables $x, y, z, \cdots$.

It may fail to be a statement.

However, provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

The key examples of predicates in school mathematics are:

- Equations with one unknown (or more).
- Inequalities with one unknown (or more).

We can also construct a statement out of a predicate by the use of quantifiers which eliminate the ambiguity due to the presence of the variables.

5. **Equations and inequalities as predicates.**

(a) Every **equation** with one unknown (or more) is a predicate in which the variables are the unknowns of the equation.

Examples.

(i) $x^2 - 1 = 0$.

(ii) $x^2 + 1 = 0$.

(iii) $x + 2y + 3 = 0$.

(iv) $x^2 + y^2 = 1$.

(v) $x^2 - y^2 = 1$.

(vi) $x + y + z = 1$.

(vii) $x^2 + y^2 + z^2 = 1$.

(viii) $x^2 + y^2 = z^2$.

(b) Every **inequality** with one unknown (or more) is a predicate in which the variables are the unknowns of the inequality.

Examples.

(i) $x^2 - 1 \geq 0$.

(ii) $x^2 + 1 > 0$.

(iii) $x + 2y + 3 \leq 0$.

(iv) $x^2 + y^2 < 1$.

(v) $x^2 - y^2 < 1$.

(vi) $x + y + z \geq 1$.

(vii) $x^2 + y^2 + z^2 \leq 1$.

(viii) $x^2 + y^2 \leq z^2$.

6. **What is 'solving an equation/inequality'?**

To **solve** an equation/inequality with unknowns $x, y, z, \cdots$ amongst so-and-so is to specify, for that equation/inequality regarded as a predicate with variables $x, y, z, \cdots$, all the 'concrete objects' amongst so-and-so which, upon 'substitution into the variables' of the predicate, turn the predicate into a true statement. Each such 'concrete object' which turn the predicate into a true statement is called a **solution** for that equation/inequality.

In practice, this is what we usually do:

- First perform some manipulation, starting from the equation/inequality concerned, in order to find all possible candidates for $x, y, z, \cdots$.
- Then substitute these candidates for $x, y, z, \cdots$ into the predicate (which is the equation/inequality concerned) to see whether we obtain a true statement.

Illustration:
What do we mean by
'Solve
$x^2 - 3x + 2 = 0$ — $(*)$
(with unknown x amongst the reals)'?

'Manipulation':
$x^2 - 3x + 2 = 0$.
$(x-1)(x-2) = 0$.
$x - 1 = 0$ or $x - 2 = 0$.
$x = 1$ or $x = 2$.

Have we finished?
• All we have done is saying 'if $x^2-3x+2=0$ then ($x=1$ or $x=2$)'.
This is not enough.

'Checking': If $x=1$ then $x^2-3x+2=\ldots=0$.
If $x=2$ then $x^2-3x+2=\ldots=0$.

Hence the solution of $(*)$ is '$x=1$ or $x=2$'.

7. Further Examples on mathematical statements and predicates.

(a) *For any $n \in \mathbb{N}$, $n^2 + n$ is an even integer.*

We are used to read it in school maths as:

Let $n \in \mathbb{N}$. n^2+n is an even integer.

Remark. $\mathbb{N}$ 'starts from' 0 in this course.

(b) **Bernoulli's Inequality.**

For any $m \in \mathbb{N}\backslash\{0,1\}$, for any $\beta \in (-1,+\infty)\backslash\{0\}$, $(1+\beta)^m > 1 + m\beta$.

We may read it as:

Let $m \in \mathbb{N}\backslash\{0,1\}$. Let $\beta \in (-1,+\infty)\backslash\{0\}$. $(1+\beta)^m > 1+m\beta$.

(c) **Mean-Value Theorem**.

Assumption

Let $a, b \in \mathbb{R}$. Suppose $a < b$.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[a, b]$ and f is differentiable on (a, b).

Then there exists some $c \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(c)$.

Conclusion

This indicates the presence of an existential quantifier

In plain language, this reads:
Somewhere amongst the numbers in the interval (a, b), there is something which we label as c, for which '$f(b)-f(a) = (b-a) f'(c)$' holds.

On its own, '$f(b)-f(a) = (b-a) f'(c)$' is a predicate with variable c.

(d) **Definition of differentiability**.

Let I be an open interval, and $c \in I$.

Let $f : I \longrightarrow \mathbb{R}$ be a function.

f is said to be differentiable at c if the limit $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists in $\mathbb{R}$.

This indicates we are reading a definition.

This is understood as 'if and only if'.
(This is a convention for definitions in this course.)

'Defining condition', explaining what 'being differentiable at a point' means in precise mathematical terms.

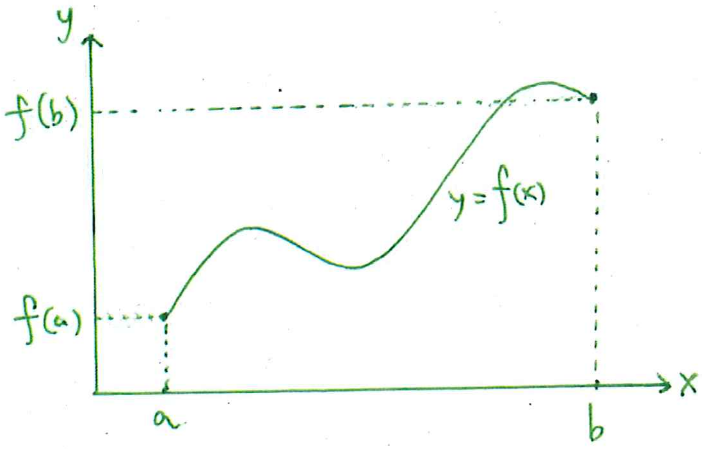
Mean Value Theorem.

Assumptions:

- $a, b \in \mathbb{R}$. $a < b$
- $f : [a, b] \to \mathbb{R}$ is a function.

  f is continuous on $[a, b]$.

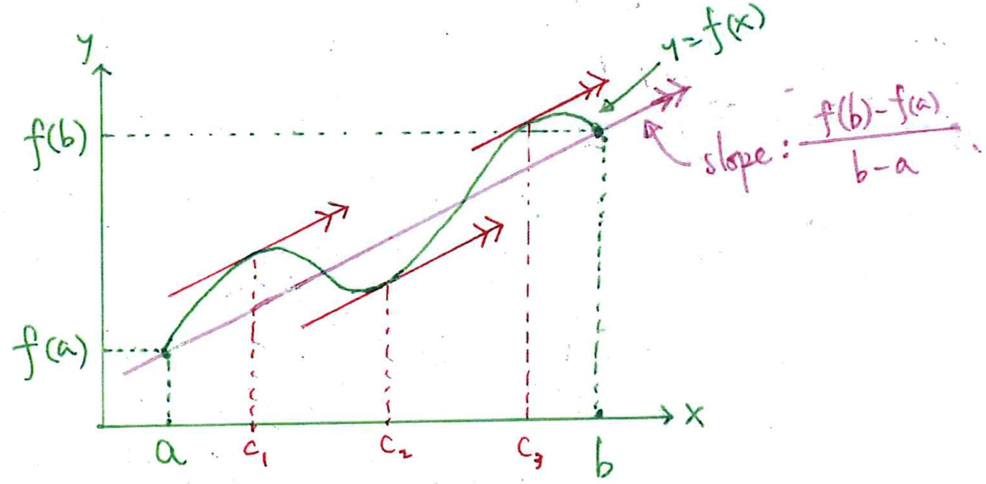
  f is differentiable on (a, b).



Conclusion:

- There exists some $c \in (a, b)$ such that

  $f(b) - f(a) = (b-a) f'(c)$.



(e) **Definition of divisibility.**

'Defining condition', explaining what 'being divisible' means in precise mathematical terms.

Let $a, b \in \mathbb{Z}$.

a is said to be divisible by b if there exists some $k \in \mathbb{Z}$ such that $a = kb$.

Indicator that we are talking about a definition.

In plain language, this reads:
Somewhere amongst the integers, there is something which we label as k for convenience, for which '$a = kb$' is true.
It does not matter whether we know how to write down the value of such a k or not.

(f) **Definition of even-ness and odd-ness for integers.**

Let $n \in \mathbb{Z}$.

* *n is said to be even if n is divisible by 2.*
* *n is said to be odd if n is not divisible by 2.*

We are defining 'even-ness', 'odd-ness' in terms of divisibility.

So if you do not understand the definition of divisibility correctly, you will not get 'even-ness', 'oddness' correctly.

(g) **Definition of prime numbers.**

Let $p \in \mathbb{Z}$. Suppose $|p| \geq 2$.

p is said to be a prime number if the following condition holds:

- $\boxed{\text{for any } k \in \mathbb{Z}, \text{ if } p \text{ is divisible by } k \text{ then } (k = 1 \text{ or } k = -1 \text{ or } k = p \text{ or } k = -p).}$

This as a whole gives the meaning of 'being a prime number' in precise mathematical terms.

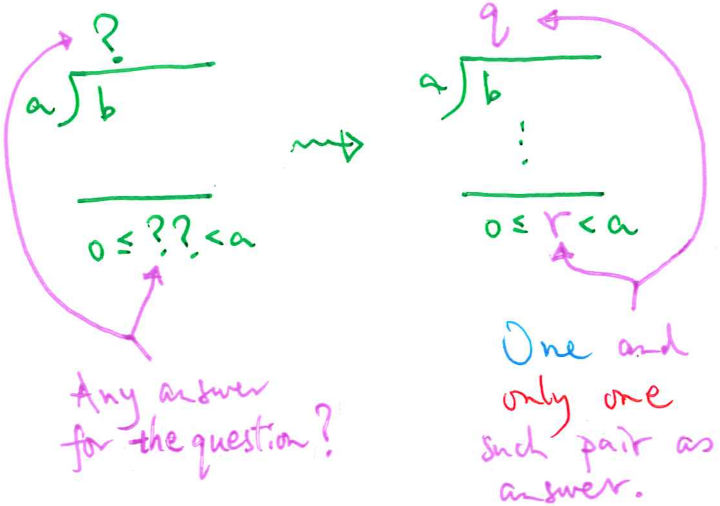
We are defining 'being a prime number' in terms of 'divisibility'.

More 'condensed' formulation:
'p is divisible by no integer other than $1, -1, p, -p$.'

(h) **Division Algorithm (for natural numbers).**

Let $a, b \in \mathbb{N}$. Suppose $a \neq 0$.

Then there exist some unique $q, r \in \mathbb{N}$ such that $b = qa + r$ and $0 \leq r < a$.



This is a very 'condensed' way of writing:

Let $a, b \in \mathbb{N}$. Suppose $a \neq 0$. Then both of ①, ② are true:

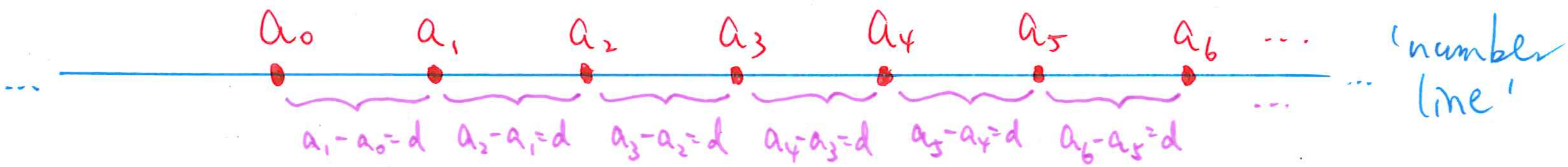
① There exist some $q, r \in \mathbb{N}$ such that $b = qa + r$ and $0 \leq r < a$.

② Suppose $q, r, q', r' \in \mathbb{N}$.
Further suppose $b = qa + r$, and $b = q'a + r'$, and $0 \leq r < a$, and $0 \leq r' < a$.
Then $q = q'$ and $r = r'$.

(i) Definition of arithmetic progression.

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence of numbers.
$\{a_n\}_{n=0}^{\infty}$ is said to form an arithmetic progression if there exists some number d such that for any $n \in \mathbb{N}$, $a_{n+1} - a_n = d$.
The number d is called the common difference of this arithmetic progression.

This is nothing but the old story which you have learnt:



(j) Definition of geometric progression.

Let $\{b_n\}_{n=0}^{\infty}$ be an infinite sequence of non-zero numbers.
$\{b_n\}_{n=0}^{\infty}$ is said to form a geometric progression if there exists some non-zero number r such that for any $n \in \mathbb{N}$, $b_{n+1}/b_n = r$.
The number r is called the common ratio of this geometric progression.