Tutorial 11 (Nov 17)

1. Show that if a function $f \in C(\mathbb{R})$ can be uniformly approximated on $\mathbb{R}$ by polynomials, then f is itself a polynomial.

2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)e^{nx}dx = 0 \quad \text{for all } n = 0, 1, 2, \ldots$$

   Show that $f \equiv 0$.

3. **Notice:** I am sorry that I made a mistake in the definition of a Baire space in the tutorial. The following is the correct definition.

   Definition: A Baire space is a topological space X that satisfies any one of the following equivalent properties:

   (a) ~~Every countable union of nowhere dense closed subsets of X is nowhere dense in X.~~

   Every countable union of closed subsets of X with empty interior has empty interior.

   (b) Every countable union of dense open subsets of X is dense in X.

   The Baire Category Theorem can be reformulated as follow:

   Theorem (Baire Category Theorem). *Every complete metric space is a Baire space.*

4. (a) Show that $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ are Baire spaces.

   (b) Show that $\mathbb{Q}$ is not a Baire space.

5. (a) Show that any open subset of a Baire space is a Baire space.

   (b) Give an example to show that a closed subset of a Baire space may not be a Baire space. (You may consider the sets $\mathbb{R}^2 \setminus ((\mathbb{R} \setminus \mathbb{Q}) \times \{0\})$ and $\mathbb{Q} \times \{0\}$.)

6. Let $f : X \to \mathbb{R}$ be the pointwise limit of a sequence of continuous functions $f_n : X \to \mathbb{R}$. Show that the set of point of continuity of f is dense in X.