MATH5011 Exercise 9

(1) Optional. Let $\mathfrak{M}$ be the collection of all sets E in the unit interval $[0,1]$ such that either E or its complement is at most countable. Let μ be the counting measure on this σ-algebra $\mathfrak{M}$. If $g(x) = x$ for $0 \leq x \leq 1$, show that g is not $\mathfrak{M}$-measurable, although the mapping

$$f \mapsto \sum x f(x) = \int f g \, d\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \neq L^\infty$ in this situation.

Solution: g is not $\mathfrak{M}$-measurable because $g^{-1}\left(\frac{1}{4}, \frac{3}{4}\right) = \left(\frac{1}{4}, \frac{3}{4}\right) \notin \mathfrak{M}$. The functional $\Lambda f = \sum x f(x)$ is clearly linear. To see that it is bounded, if $f \in L^1(\mu)$, then f is non-zero on an at most countable set $\{x_i\}$ and by integrability,

$$\sum_{i=1} |f(x_i)| < \infty.$$

Thus Λf is well defined as g is a bounded function. Hence the operator is bounded.

(2) Optional. Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0,1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$, and therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I f g \, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

Solution: Method 1. For any $x \in I$ take $\Lambda_x f = g(x_+) - g(x_-)$ for all f such that $f = g$ a.e. for some function g such that the two one-sided limits $g(x_+)$ and $g(x_-)$ both exist. Then $\|\Lambda_x - \Lambda_y\| \geq 1$ for $x \neq y$. With reference to the question, we can just take $x = 1/2$.

Method 2. Consider $\chi_{[0,\frac{1}{2}]} \in L^\infty \setminus C(I)$, as $C(I)$ is closed subspace in L^∞, by consequence of Hahn-Banach Theorem (thm 3.11 in p.38 of lecture notes on functional analysis.), there is non-zero bounded linear functional Λ on L^∞ which is zero on $C(I)$.

If there is $g \in L^1(m)$ that satisfies $\Lambda f = \displaystyle\int_I fg\,dm$ for every $f \in L^\infty$,

$$\Lambda f = \int_I fg\,dm = 0, \forall f \in C(I) \Rightarrow g = 0.$$

we have $\Lambda = 0$ which is impossible.

(3) Prove Brezis-Lieb lemma for $0 < p \leq 1$.

Hint: Use $|a+b|^p \leq |a|^p + |b|^p$ in this range.

Solution: Taking $g_n = f_n - f$ as a and f as b,

$$\left| \, |f + g_n|^p - |g_n|^p \, \right| \leq |f|^p \, ,$$

or,

$$- |f|^p \leq |f + g_n|^p - |g_n|^p \leq |f|^p \, .$$

we have

$$-2 \, |f|^p \leq |f + g_n|^p - |g_n|^p - |f|^p \leq 0$$

which implies

$$\left| \, |f + g_n|^p - |g_n|^p - |f|^p \, \right| \leq 2 \, |f|^p \, ,$$

and result follows from Lebesgue dominated convergence theorem.

(4) Let $f_n, f \in L^p(\mu)$, $0 < p < \infty$, $f_n \to f$ a.e., $\|f_n\|_p \to \|f\|_p$. Show that $\|f_n - f\|_p \to 0$.

Solution: Using the Brezis-Lieb lemma for $0 < p < \infty$, we have

$$\begin{aligned}\|f_n - f\|_p^p &= \int_X |f_n - f|^p \; d\mu \\ &\leq \int_X (|f_n - f|^p - (|f_n|^p - |f|^p)) \; d\mu + \int_X (|f_n|^p - |f|^p) \; d\mu \\ &\leq \int_X ||f_n - f|^p - (|f_n|^p - |f|^p)| \; d\mu + \left(\|f_n\|_p^p - \|f\|_p^p\right) \\ &\to 0\end{aligned}$$

as $n \to \infty$.

(5) Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \ldots$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p \, d\mu < C$ for all n. Prove that

$$\lim_{n\to\infty} \int_X |f - f_n| \, d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

Solution: By Vitali's convergence Theorem, it suffices to prove that $\{f_n\}$ is uniformly integrable. Let q be conjugate to p. By Hölder inequality,

$$\begin{aligned}\int_E |f_n| d\mu &\leq \|f_n\|_p \{\mu(E)\}^{\frac{1}{q}} \\ &\leq C^{\frac{1}{p}} \{\mu(E)\}^{\frac{1}{q}},\end{aligned}$$

for any measurable E. Now the result follows easily.

(6) We have the following version of Vitali's convergence theorem. Let $\{f_n\} \subset L^p(\mu)$, $1 \leq p < \infty$. Then $f_n \to f$ in L^p-norm if and only if

(i) $\{f_n\}$ converges to f in measure,

(ii) $\{|f_n|^p\}$ is uniformly integrable, and

(iii) $\forall \varepsilon > 0$, $\exists$ measurable E, $\mu(E) < \infty$, such that $\displaystyle\int_{X \setminus E} |f_n|^p \, d\mu < \varepsilon$, $\forall n$.

I found this statement from PlanetMath. Prove or disprove it.

Solution: Let $\varepsilon > 0$. By (iii), there exists a set E of finite measure (WLOG assume positive measure) such that

$$\int_{\widetilde{E}} |f_n|^p < \varepsilon.$$

Since $\{f_n\}$ converges to f in measure, there is a subsequence $\{f_{n_k}\}$ which converges to f pointwisely a.e.. By Fatou's Lemma,

$$\int_{\widetilde{E}} |f|^p < \varepsilon.$$

By (ii), there exists $\delta > 0$ such that whenever $\mu(A) < \delta$,

$$\int_A |f_n|^p < \varepsilon^{\frac{1}{p}};$$

WLOG, by choosing a smaller δ, we may assume further whenever $\mu(A) < \delta$

$$\int_A |f|^p < \varepsilon^{\frac{1}{p}}$$

because there is a subsequence $\{f_{n_k}\}$ which converges to f pointwisely a.e. and we can apply Fatou's Lemma, By (i), there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mu\{x \in E : \Big|(f_n - f)(x)\Big|^p \geq \frac{\varepsilon}{\mu(E)}|\} < \delta.$$

Now, for $n \geq \mathbb{N}$, define $A_n = \{x \in E : |(f_n - f)(x)|^p \geq \frac{\varepsilon}{\mu(E)}\}$ and $B_n = E \setminus A_n$, and we have

$$\begin{aligned}
\int |f_n - f|^p &= \int_{\widetilde{E}} |f_n - f|^p + \int_E |f_n - f|^p \\
&< 2^p \varepsilon + \int_{A_n} |f_n - f|^p + \int_{B_n} |f_n - f|^p \\
&< 2^p \varepsilon + \left(\int_{A_n} |f_n|^p + \int_{A_n} |f|^p \right)^p + \varepsilon \\
&< 2^p \varepsilon + 2^p \varepsilon + \varepsilon = (2^{p+1} + 1)\varepsilon.
\end{aligned}$$

This completes the proof.

(7) Let $\{x_n\}$ be bounded in some normed space X. Suppose for Y dense in X', $\Lambda x_n \to \Lambda x$, $\forall \Lambda \in Y$ for some x. Deduce that $x_n \rightharpoonup x$.

Solution: Since $\{x_n\}$ is bounded, there exists $M > 0$ such that $\|x_n\| \le M$. Write $M_1 = \max\{M, \|x\|\}$.

Given $\varepsilon > 0$ and $\Lambda \in X'$, choose $\Lambda_1 \in Y$ such that $\|\Lambda - \Lambda_1\| < \dfrac{\varepsilon}{3M_1}$ and choose N large such that $|\Lambda x_n - \Lambda x| < \dfrac{\varepsilon}{3}$. Then

$$\begin{aligned}
|\Lambda x_n - \Lambda x| &= |\Lambda x_n - \Lambda_1 x_n| + |\Lambda_1 x_n - \Lambda_1 x| + |\Lambda_1 x - \Lambda x| \\
&\le \frac{\varepsilon}{3M_1} M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M_1} \|x\| \\
&< \varepsilon.
\end{aligned}$$

(8) Consider $f_n(x) = n^{1/p} \chi(nx)$ in $L^p(\mathbb{R})$. Then $f_n \rightharpoonup 0$ for $p > 1$ but not for $p = 1$. Here $\chi = \chi_{[0,1]}$.

Solution: For $1 < p < \infty$, let q be the conjugate exponent and let $g \in L^q(\mathbb{R})$.

By Hölder's inequality and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\int_{\mathbb{R}} f_n g\,dx &= \int_0^{\frac{1}{n}} n^{1/p} g(x)\,dx \\ &\leq \left(\int_0^{\frac{1}{n}} (n^{1/p})^p\,dx\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{n}} |g(x)|^q\,dx\right)^{\frac{1}{q}} \\ &\leq \left(\int_{\mathbb{R}} \chi_{[0,\frac{1}{n}]} |g(x)|^q\,dx\right)^{\frac{1}{q}} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Hence, $f_n \rightharpoonup 0$.

For $p = 1$, take $g \equiv 1$ in $L^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} f_n g\,dx = n \int_0^{\frac{1}{n}} \,dx = 1.$$

Hence, $f_n \not\rightharpoonup 0$.

(9) Let $\{f_n\}$ be bounded in $L^p(\mu)$, $1 < p < \infty$. Prove that if $f_n \to f$ a.e., then $f_n \rightharpoonup f$. Is this result still true when $p = 1$?

Solution: It suffices to show that for any $g \in L^q(\mu)$,

$$\int (f_n - f) g d\mu \to 0 \text{ as } n \to \infty.$$

By Prop 4.14 the density theorem, we may consider the case where g is a simple function with finite support. Let E be a finite measure set such that $g = 0$ outside E and $M > 0$ be bound of g. By the solution to Problem 5, $\{f_n, f\}$ is uniformly integrable, for all $\varepsilon > 0, \exists \delta > 0$, s.t. for any A measurable s.t $\mu(A) < \delta$,

$$\int_A |h| d\mu < \varepsilon, h = f_n \text{ or } f.$$

By Egorov's Theorem, there is a measurable B s.t $\mu(E \setminus B) < \delta$ and f_n

converges uniformly to f on B. Hence

$$\begin{aligned}\left|\int (f_n - f) g d\mu\right| &= \left|\int_E (f_n - f) g d\mu\right| \\ &= \left|\int_{E\setminus B} (f_n - f) g d\mu\right| + \left|\int_B (f_n - f) g d\mu\right| \\ &< 2M\varepsilon + \left|\int_B (f_n - f) g d\mu\right| \\ &< (2M+1)\varepsilon, \text{ for large n }.\end{aligned}$$

For p=1, the result is false by Problem 8.

(10) Provide a proof of Proposition 5.3.

Solution:

(a) Let $E = \overset{\circ}{\bigcup} E_j \in \mathfrak{M}$. If λ is concentrated on A, then $\lambda(E_j) = \lambda(E_j \cap A)$, and so

$$\begin{aligned}|\lambda|(E) &= \sup\{\sum |\lambda(E_j)| : \ E = \overset{\circ}{\bigcup} E_j, \ E_j \in \mathfrak{M}\} \\ &= \sup\{\sum |\lambda(E_j \cap A)| : \ E \cap A = \overset{\circ}{\bigcup} (E_j \cap A), \ E_j \in \mathfrak{M}\} \\ &= |\lambda|(E \cap A).\end{aligned}$$

(b) If $\lambda_1 \perp \lambda_2$, then λ_j is concentrated on some A_j $(j = 1, 2)$ with $A_1 \cap A_2 = \emptyset$. By part (a), $|\lambda_j|$ is concentrated on A_j. Therefore, $|\lambda_1| \perp |\lambda_2|$.

(c) Suppose μ is concentrated on A. If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1(A) = \lambda_2(A) = 0$, which implies $(\lambda_1 + \lambda_2)(A) = 0$. Hence, $\lambda_1 + \lambda_2 \perp \mu$.

(d) Suppose $\mu(E) = 0$. If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1(E) = \lambda_2(E) = 0$, which implies $(\lambda_1 + \lambda_2)(E) = 0$. Hence, $\lambda_1 + \lambda_2 \ll \mu$.

(e) Let $E = \overset{\circ}{\bigcup} E_j$ and suppose $\mu(E) = 0$. Then $E_j \subset E$ implies $\mu(E_j) = 0$. If $\lambda \ll \mu$, then $\lambda(E_j) = 0$. Therefore, $\sum |\lambda(E_j)| = 0$ and it follows that $|\lambda|(E) = 0$.

(f) Suppose λ_2 is concentrated on A. If $\lambda_2 \perp \mu$, then $\mu(A) = 0$, which implies $\lambda_1(A) = 0$ by $\lambda_1 \ll \mu$. Hence, $\lambda_1 \perp \lambda_2$.

(g) By part (f), $\lambda \perp \lambda$. This is impossible unless $\lambda = 0$.

(11) Show that $M(X)$, the space of all signed measures defined on $(X, \mathfrak{M})$, forms a Banach space under the norm $\|\mu\| = |\mu|(X)$.

Solution: It is clear that the $M(X)$ is a normed vector space if the norm is defined as in the question.

Recall the fact that a normed vector space is a Banach space if and only if every absolutely summable sequence is summable. Let $\{\mu_k\}$ be an absolutely summable sequence. Let E be a measurable set. We immediately have

$$\sum_{k=1}^{\infty} |\mu_k(E)| \leq \sum_{k=1}^{\infty} |\mu_k|(E) \leq \sum_{k=1}^{\infty} |\mu_k|(X) < \infty,$$

hence $\sum \mu_k(E)$ converges absolutely. $\forall E \in \mathfrak{M}$, put

$$\mu(E) = \sum_{k=1}^{\infty} \mu_k(E)$$

which exists as a real number by the above argument. We will prove the countable additivity. Let E_n be a sequence of pairwise disjoint measurable sets. Then

$$\begin{aligned}
\mu\left(\bigcup E_n\right) &= \sum_{k=1}^{\infty} \mu_k \left(\bigcup E_n\right) \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mu_k(E_n) \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu_k(E_n) \text{ (by absolute convergence)} \\
&= \sum_{n=1}^{\infty} \mu(E_n).
\end{aligned}$$

We have proved that μ is a signed measure. To show that μ_n converges to μ in $\|\cdot\|$, let X_n be a partition of X.

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\left(\mu-\sum_{k=1}^{m}\mu_k\right)(X_n)\right| &= \left|\sum_{n=1}^{\infty}\sum_{k=m}^{\infty}\mu_k(X_n)\right| \\ &\leq \sum_{k=m}^{\infty}\sum_{n=1}^{\infty}|\mu_k(X_n)| \\ &\leq \sum_{k=m}^{\infty}|\mu_k|(X)=\sum_{k=m}^{\infty}\|\mu_k\|\to 0\end{aligned}$$

so that $\left\|\sum \mu_k-\mu\right\|\to 0$ as $k\to\infty$.

(12) Let $\mathcal{L}^1$ be the Lebesgue measure on $(0,1)$ and μ the counting measure on $(0,1)$. Show that $\mathcal{L}^1\ll\mu$ but there is no $h\in L^1(\mu)$ such that $d\mathcal{L}^1=h\,d\mu$. Why?

Solution: If $\mu(E)=0$, then $E=\phi$, which implies $\mathcal{L}^1(E)=0$. Hence, $\mathcal{L}^1\ll\mu$.

Suppose on the contrary, that $\exists h\in L^1(\mu)$ such that $d\mathcal{L}^1=\int h\,d\mu$. Since $h\in L^1(\mu)$, $h=0$ except on a countable set. It follows that $\mathcal{L}^1(\{h=0\})=1$. However,

$$\mathcal{L}^1(\{h=0\})=\int_{\{h=0\}}h\,d\mu=0.$$

This is a contradiction. Radon-Nikodym theorem does not apply here because μ is not σ-finite.

(13) Let μ be a measure and λ a signed measure on $(X,\mathfrak{M})$. Show that $\lambda\ll\mu$ if and only if $\forall\varepsilon>0$, there is some $\delta>0$ such that $|\lambda(E)|<\varepsilon$ whenever $|\mu(E)|<\delta$, $\forall E\in\mathfrak{M}$.

Solution: ($\Leftarrow$) Suppose $\mu(E)=0$. By the hypothesis, for all $\varepsilon>0$, $|\lambda(E)|<\varepsilon$. This implies $\lambda(E)=0$, hence $\lambda\ll\mu$.

($\Rightarrow$) Suppose on the contrary that $\exists\varepsilon_0>0$ such that $\forall n\in\mathbb{N}$, $\exists E\in\mathfrak{M}$ with

$\mu(E) < 2^{-n}$ such that $\lambda(E) < \varepsilon$. Put $E = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$. Then $\mu(E) = 0$ but

$$\lambda(E) = \lim_{n\to\infty} \lambda\left(\bigcup_{k\geq n} E_k\right) \geq \varepsilon_0 > 0.$$

This contradicts the fact that $\lambda \ll \mu$.

(14) Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Show that

$$\int f\, d\lambda = \int fh\, d\mu, \quad \forall f \in L^1(\lambda),\ fh \in L^1(\mu)$$

where $h = \dfrac{d\lambda}{d\mu} \in L^1(\mu)$.

Solution

Step 1. $f = \chi_E$ for some $E \in \mathfrak{M}$.

We have

$$\int_X \chi_E\, d\lambda = \lambda(E) = \int_E h\, d\mu = \int_X \chi_E h\, d\mu.$$

Step 2. f is a simple function.

This follows directly from Step 1.

Step 3. $f \geq 0$ is measurable.

Pick $0 \leq s_n \nearrow f$. Then $0 \leq s_n h \nearrow fh$ on $\{h \geq 0\}$ and $0 \leq -s_n h \nearrow$

$-fh$ on $\{h < 0\}$. Hence,

$$\begin{aligned}
\int_X f\, d\lambda &= \int_{h\geq 0} f\, d\lambda - \int_{h<0} -f\, d\lambda \\
&= \sup_{0\leq s\leq f} \int_{h\geq 0} s\, d\lambda - \sup_{0\leq s\leq f} \int_{h<0} -s\, d\lambda \\
&= \sup_{0\leq s\leq f} \int_{h\geq 0} sh\, d\mu - \sup_{0\leq s\leq f} \int_{h<0} -sh\, d\mu \text{ (by Step 2)} \\
&= \int_{h\geq 0} fh_+\, d\mu - \int_{h<0} fh_-\, d\mu \\
&= \int_X f(h_+ - h_-)\, d\mu \\
&= \int_X fh\, d\mu.
\end{aligned}$$

Step 4. $f \in L^1(\lambda)$.

Writing $f = f_+ - f_-$, the result follows from Step 3.

(15) Let μ, λ and ν be finite measures, $\mu \gg \lambda \gg \nu$. Show that $\dfrac{d\nu}{d\mu} = \dfrac{d\nu}{d\lambda}\dfrac{d\lambda}{d\mu}$, μ a.e.

Solution: By (14), we have for all measurable sets E,

$$\nu(E) = \int_E \frac{d\nu}{d\lambda}\, d\lambda = \int_E \frac{d\nu}{d\lambda}\frac{d\lambda}{d\mu}\, d\mu.$$

The result follows from the uniqueness of the Radon-Nikodym derivative.