# MATH5011 Exercise 7

For those who have learnt functional analysis, this exercise serves to refresh your memory. For those who have not learnt it, working through the problems gives you some feeling on the subject.

(1) Provide two proofs that $C[0,1]$ is an infinite dimensional vector space.

(2) Show that both $C_c(0,1)$ and $C^1[0,1]$ are not closed subspaces in $C[0,1]$ and hence they are not Banach spaces.

(3) Endow $C[0,1]$ with the norm $\|f\| = \int_0^1 |f(x)|dx$. Determine whether it is complete or not.

(4) Let $\Lambda$ be a bounded linear functional on the normed space $X$. Show that its operator norm

$$\begin{aligned}\|\Lambda\|_{op} &= \sup\left\{\frac{\Lambda x}{\|x\|} : x \neq 0\right\}\\ &= \inf\left\{M : |\Lambda x| \leq M\|x\|\ ,\ \forall x \in X\right\}.\end{aligned}$$

(5) For any normed space $(X, \|\cdot\|$, prove that $(X', \|\cdot\|_{op})$ forms a Banach space.

(6) Let $X$ be a Hilbert space and $X_1$ a proper closed subspace. For $x_0$ lying outside $X_1$, let $d = \|x_0 - z\|$ where $d$ is the distance from $x_0$ to $X_1$. Show that

$$\langle x, z - x_0\rangle = 0, \quad \forall x \in X_1.$$

Hint: For $x \in X_1$, one has $\dfrac{d}{dt}\phi(t) = 0$ at $t = 0$ where $\phi(t) = \|z_0 + tx - x_0\|^2$. Why?

(7) Show that the correspondence $\Lambda \mapsto w$ in Theorem 4.8 is norm preserving.

(8) Let $\Lambda_1$ and $\Lambda_2$ be two bounded linear functionals on the Hilbert space $X$. Suppose that they have the same kernel. Prove that there exists a nonzero constant $c$ such that $\Lambda_2 = c\Lambda_1$. Use this fact to give a proof of Theorem 4.8.

(9) This is optional. Read Page 23 and on in [SS] for the following striking application of the Hahn-Banach theorem:

There exists $m : \mathcal{P}_{\mathbb{R}} \to [0, \infty]$ satisfying

(1) $m(E_1 \cup E_2) = m(E_1) + m(E_2)$, $E_1, E_2 \subset \mathbb{R}$, $E_1 \cap E_2 = \emptyset$,

(2) $m(E) = \mathcal{L}^1(E)$ whenever $E$ is $\mathcal{L}^1$-measurable,

(3) $m(E + a) = m(E)$, $\forall E \subset \mathbb{R}$, $\forall a \in \mathbb{R}$.

Of course, $m$ cannot be countably additive.