MATH5011 Real Analysis I

Exercise 2

Notations in lecture notes are used.

(1) Let f be a non-negative measurable function.

(a) Prove Markov's inequality:

$$\mu\Big\{x \in X: \ f(x) \geq M\Big\} \leq \frac{1}{M}\int_X f d\mu,$$

for all $M > 0$.

(b) Deduce that every integrable function is finite a.e..

(c) Deduce that $f = 0$ a.e. if f is integrable and $\int f = 0$.

(2) Let g be a measurable function in $[0, \infty]$. Show that

$$m(E) = \int_E g \, d\mu$$

defines a measure on $\mathcal{M}$. Moreover,

$$\int_X f \, dm = \int_X fg \, d\mu, \qquad \forall f \text{ measurable in } [0, \infty].$$

(3) Let $\{f_k\}$ be measurable in $[0, \infty]$ and $f_k \searrow f$ a.e., f measurable and $\displaystyle\int f_1 \, d\mu < \infty$. Show that

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

What happens if $\displaystyle\int f_1 \, d\mu = \infty$?

(4) Let f be a measurable function. Show that there exists a sequence of simple functions $\{s_j\}$, $|s_1| \leq |s_2| \leq |s_3| \leq \cdots$, and $s_k(x) \to f(x)$, $\forall x \in X$.

(5) Let $\mu(X) < \infty$ and f be integrable. Suppose that

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in [a, b], \ \forall E \in \mathfrak{M}, \mu(E) > 0$$

for some $[a, b]$. Show that $f(x) \in [a, b]$ a.e.

(6) Let f be Lebsegue integrable on $[a, b]$ which satisfies

$$\int_a^c f d\mathcal{L}^1 = 0,$$

for every c. Show that f is equal to 0 a.e..

(7) Let $f \geq 0$ be integrable and $\displaystyle\int f \, d\mu = c \in (0, \infty)$. Prove that

$$\lim_{n\to\infty} \int n \log \left(1 + \left(\frac{f}{n}\right)^\alpha\right) \, d\mu = \begin{cases} \infty, & \text{if } \alpha \in (0,1) \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

(8) Let f be a non-negative integrable function with respect to some μ and let $F_k = \{x : f(x) \geq k\}$ for $k \geq 1$. Show that $\sum_k \mu(F_k) < \infty$. Hint: Relate F_k to $E_k = \{x : k \leq f(x) \leq k + 1\}$.

(9) Let $\mu(X) < \infty$ and $f_k \to f$ uniformly on X and each f_k is bounded. Prove that

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

Can $\mu(X) < \infty$ be removed?

(10) Give another proof of Borel-Cantelli lemma in Ex.1 by using Corollary 1.12. Hint: Study $g(x) = \displaystyle\sum_{j=1}^{\infty} \chi_{A_j}(x)$.

(11) Give an example of a sequence $\{f_k\}$ on $[0, 1]$, $f_k \to f$ in L^1 with respect to $\mathcal{L}^1$ but it does not converge at any point in $[0, 1]$.

Hint: Divide $[0,1]$ into $2^k, k \geq 1$, many subintervals of equal length and order them in a sequence. Let $I_j^k, j = 1, 2, \cdots, 2^k$, be these subintervals and consider the sequence composed of the characteristic functions of I_j^k.

(12) Let f be a Riemann integrable function on $[a,b]$ and extend it to $\mathbb{R}$ by setting it zero outside $[a,b]$.

(a) Show that f is Lebsegue measurable.

(b) Show that the Riemann integral of f is equal to $\int_{\mathbb{R}} f d\mathcal{L}^1$.

(c) Give an example of a sequence of Riemann integrable functions which is uniformly bounded on $[a,b]$ and converges pointwisely to some function which is not Riemann integrable.