Chapter 2

Outer Measures

There are two ways to construct measure spaces, namely, Caratheodory's approach via outer measures and Riesz representation theorem. In Section 1 we discuss outer measures and show there is always a measure space associated to an outer measure. Outer measures require less stringent conditions than measures, so they are easier to construct. Next we study Borel measures. These are measures in which continuous functions are measurable. We will work on a topological space. Basic notions in point set topology and metric spaces are reviewed and, Borel sets, the σ-algebra generated by open sets, are discussed in Section 2. Sections 3 and 4 are devoted to the Riesz representation theorem. The setting is a locally compact Hausdorff topological space where continuous functions are abundant. We prove the Riesz representation theorem by constructing an outer measure for every positive functional. In Section 5 we prove Lusin's theorem, which is about how to approximate measurable functions by continuous functions. Finally, in Section 6 we briefly discuss two classes of special measurable function with respect to a Borel measure. These functions are often used to approximate a general measurable function in various context.

2.1 Outer Measures

Let X be a non-empty set. An *outer measure* on X is a function μ from $\mathcal{P}_X$ to $[0,\infty]$ satisfying

(i) $\mu(\phi) = 0$,

(ii) (countable subadditivity)

$$\mu(A) \le \sum_{j=1}^{\infty} \mu(A_j) \ ,$$

whenever $A \subset \cup_{j=1}^{\infty} A_j, \ A_j \subset X, j \ge 1$.

Taking $A_1 = B$ and $A_j = \phi, j \geq 2$ in (ii), we have $\mu(A) \leq \mu(B)$ whenever $A \subset B$. A well-known outer measure is the Lebesgue measure on $\mathbb{R}$, which is defined by

$$\mathcal{L}^1(A) = \inf\left\{\sum_{j=1}^{\infty} |I_j| : \text{ All } \{I_j\} \text{ satisfying } A \subset \cup_j I_j\right\},$$

where $I_j = [a_j, b_j], j \geq 1$, are closed intervals and $|I_j| = b_j - a_j$. It is readily verified that $\mathcal{L}^1$ is an outer measure on $\mathbb{R}$. The construction of the Lebesgue is a typical one. It is worthwhile to generalize it in the following way. We call the pair $(\mathcal{G}, \varphi)$ where $\mathcal{G} \subset \mathcal{P}_X$ and $\varphi : \mathcal{G} \to [0, \infty]$ a "gauge" if

(a) $\inf_{G \in \mathcal{G}} \varphi(G) = 0$, and

(b) $\bigcup_j G_j = X$, for some $\{G_j\} \subset \mathcal{G}$.

With a gauge, an outer measure can be defined as follows

$$\mu(A) = \inf\left\{\sum_{j=1}^{\infty} \varphi(G_j) : \ A \subset \cup_{j=1}^{\infty} G_j, \ G_j \in \mathcal{G}\right\}.$$

To see how (i) is satisfied, observe that the empty set ϕ is contained in any G, so by (a) $\mu(\phi) = 0$. Next, by (b) every set A can be covered by the countable union of some G_j in $\mathcal{G}$, so the set for which the infimum is taken is non-empty and $\mu(A)$ is well-defined. Let $A \subset \bigcup_{j=1}^{\infty} A_j$. Suppose that $\sum_j \mu(A_j) < \infty$ (otherwise there is nothing to prove). For each $\varepsilon > 0$, we can find $G_k^j, k \geq 1$, in $\mathcal{G}$ such that $A_j \subset \bigcup_k G_k^j$ and $\sum_k \varphi(G_k^j) \leq \mu(A_j) + \varepsilon/2^j$. As $\{G_k^j\}$ covers A, we have

$$\begin{aligned}
\mu(A) &\leq \sum_{j,k} \varphi(G_k^j) \\
&\leq \sum_j \sum_k \varphi(G_k^j) \\
&\leq \sum_j \left(\mu(A_j) + \frac{\varepsilon}{2^j}\right) \\
&\leq \sum_j \mu(A_j) + \varepsilon,
\end{aligned}$$

and (ii) holds after letting ε tend to 0.

Given an outer measure μ, call a set E *measurable w.r.t.* μ, or μ-*measurable*, or simply *measurable* when the context is clear if

$$\mu(C) = \mu(C \cap E) + \mu(C \setminus E), \quad \forall C \subset X.$$

By countable subadditivity, $\mu(C) \leq \mu(C \cap E) + \mu(C \setminus E)$ always holds. Thus to establish the measurability of E it is sufficient to show that the one-sided inequality $\mu(C) \geq \mu(C \cap E) + \mu(C \setminus E)$. Denote the collection of all measurable sets by $\mathcal{M}_C$. We know that it contains at least two elements, namely, ϕ and X. In fact, it forms an σ-algebra, and this is the content of the following theorem.

Theorem 2.1. *$\mathcal{M}_C$ is a σ-algebra and $(X, \mathcal{M}_C, \mu)$ forms a measure space.*

Here we do not distinguish μ and its restriction on $\mathcal{M}_C$. This theorem is due to Caratheodory and sometimes called Caratheodory's construction of measures. The subscript "C" in $\mathcal{M}_C$ refers to his name.

Proof. First of all, from the definition of the measurability of a set we know that the complement of E, E', is measurable whenever E is measurable. Next, we claim that $E_1 \cup E_2 \in \mathcal{M}_C$ for $E_1, E_2 \in \mathcal{M}_C$. Indeed, for $C \subset X$,

$$\begin{aligned}
\mu(C) &= \mu(C \cap E_1) + \mu(C \setminus E_1) \\
&= \mu(C \cap E_1 \cap E_2) + \mu(C \cap E_1 \setminus E_2) + \mu((C \setminus E_1) \cap E_2) + \mu((C \setminus E_1) \setminus E_2) \\
&= \mu(C \cap E_1 \cap E_2) + \mu(C \cap E_1 \cap E_2') + \mu(C \cap E_1' \cap E_2) + \mu(C \setminus (E_1 \cup E_2)) \\
&\geq \mu(C \cap (E_1 \cup E_2)) + \mu(C \setminus E_1 \cup E_2),
\end{aligned}$$

where in the last step subadditivity has been used. Our claim holds.

By induction, for any $n \geq 2$, $\bigcup_{j=1}^n E_j \in \mathcal{M}_C,\ E_j \in \mathcal{M}_C$.

Using $E_1 \bigcap E_2 = \left(E_1' \bigcup E_2'\right)'$, and so on we know that $\bigcap_{j=1}^n E_j \in \mathcal{M}_C$ for $E_j \in \mathcal{M}_C$.

Now, given $\{E_j\}, j \geq 1$, in $\mathcal{M}_C$, we want to show that $\bigcup_{j=1}^\infty E_j \in \mathcal{M}_C$. We assume E_j's are mutually disjoint first. For $C \subset X$,

$$\begin{aligned}
\mu(C \cap A_n) &= \mu(C \cap A_n \cap E_n) + \mu(C \cap A_n \setminus E_n) \\
&= \mu(C \cap E_n) + \mu(C \cap A_{n-1})
\end{aligned}$$

where

$$A_n = \bigcup_{j=1}^n E_j \in \mathcal{M}, \text{ and } A = \bigcup_{j=1}^\infty E_j.$$

Repeating n many times, we get

$$\mu(C \cap A_n) = \sum_{j=1}^n \mu(C \cap E_j), \tag{2.1}$$

Using (2.1),

$$\begin{aligned}
\mu(C) &= \mu(C \cap A_n) + \mu(C \setminus A_n) \\
&\geq \mu(C \cap A_n) + \mu(C \setminus A) \\
&= \sum_1^n \mu(C \cap E_j) + \mu(C \setminus A).
\end{aligned}$$

Letting $n \to \infty$,

$$\mu(C) \geq \sum_1^\infty \mu(C \cap E_j) + \mu(C \setminus A)$$
$$\geq \mu(C \cap A) + \mu(C \setminus A),$$

whence $A \in \mathcal{M}_C$. Taking $C = A$ in this inequality, we obtain

$$\mu(A) \geq \sum_1^\infty \mu(E_j) \quad \text{whenever } A = \bigcup_{j=1}^\infty E_j, \quad E_j \in \mathcal{M}_C, \text{ are mutually disjoint.}$$

We have shown that μ is countably additive and hence a measure on $\mathcal{M}_C$.

Finally, when $E_j \in \mathcal{M}_C$ may not be disjoint, we set $F_1 = E_1, F_2 = E_2 \setminus E_1, F_3 = E_3 \setminus \left(E_1 \bigcup E_2\right), \cdots$. Then F_j's are mutually disjoint and belong to $\mathcal{M}_C$. Using $\bigcup_j E_j = \bigcup_j F_j$, we conclude that $\bigcup_j E_j$ belongs to $\mathcal{M}_C$, so $\mathcal{M}_C$ is closed under countable union. The proof of Theorem 2.1 is completed.

□

A measure space $(X, \mathcal{M}, \mu)$ is *complete* if every subset of a null set is measurable (and hence) a null set. In Exercise 1 you are asked to show that every measure space $(X, \mathcal{M}, \mu)$ admits a completion $(X, \overline{\mathcal{M}}, \overline{\mu})$. In fact, $\overline{\mathcal{M}}$ is the σ-algebra generated by $\mathcal{M}$ and subsets N of null sets and $\overline{\mu}(A \cup N) = \overline{\mu}(A \setminus N) = \mu(A)$.

We point out that the measure space associated to an outer measure $(X, \mathcal{M}_C, \mu)$ is always complete. To see this, let B be a subset of null set A. Let us verify

$$\mu(C) \geq \mu(C \cap B) + \mu(C \setminus B), \ \forall C \subset X.$$

Indeed, from $\mu(C \cap B) \leq \mu(C \cap A)$ and $C \setminus B \subset (C \setminus A) \cup (A \setminus B)$ we have

$$\mu(C \setminus B) \leq \mu(C \setminus A) + \mu(A \setminus B)$$
$$= \mu(C \setminus A),$$

so

$$\mu(C) \geq \mu(C \cap A) + \mu(C \setminus A)$$
$$\geq \mu(C \cap B) + \mu(C \setminus B).$$

Now, an interesting question arises. Given a complete measure space $(X, \mathcal{M}, \mu)$, it is clear that $(\mathcal{M}, \mu)$ as a gauge and we can use it to define an outer measure by

$$\widetilde{\mu}(E) = \inf \left\{ \sum_j \mu(E_j) : \quad E \subset \bigcup_j E_j, \ E_j \in \mathcal{M} \right\}.$$

Let $\mathcal{M}_C$ be the σ-algebra of $\widetilde{\mu}$-measurable sets. Is $\mathcal{M}_C = \mathcal{M}$? Not quite, but we have

Theorem 2.2. *Let $(X, \mathcal{M}, \mu)$ be a complete measure space and $\widetilde{\mu}$ and $\mathcal{M}_C$ as described above.*

(a) $\mathcal{M} \subset \mathcal{M}_C$ and $\widetilde{\mu} = \mu$ on $\mathcal{M}$.

(b) $\mathcal{M} = \mathcal{M}_C$ provided $(X, \mathcal{M}, \mu)$ is σ-finite.

A measure space is *σ-finite* if there exist $X_j \in \mathcal{M},\ \mu(X_j) < \infty,\ j \geq 1$, such that $X = \bigcup_{j=1}^{\infty} X_j$. The Lebesgue measure is σ-finite on $\mathbb{R}$ as $\mathbb{R} = \bigcup_j [-j, j]$ and $\mathcal{L}^1([-j, j]) = 2j < \infty$.

Proof. First, $\widetilde{\mu}$ coincides with μ on $\mathcal{M}$. For, let $E \in \mathcal{M}$. Suppose that $E \subset \bigcup_j E_j, E_j \in \mathcal{M}$. Then

$$\mu(E) \leq \sum_j \mu(E_j)$$

by countable subadditivity. Taking supremum over all these $\{E_j\}$, we have

$$\mu(E) \leq \widetilde{\mu}(E).$$

On the other hand, from $E \subset E$ we have $\widetilde{\mu}(E) \leq \mu(E)$.

Next, for $E \in \mathcal{M}$, we wish to show

$$\widetilde{\mu}(C) \geq \widetilde{\mu}(C \cap E) + \widetilde{\mu}(C \setminus E), \quad \forall C \subset X.$$

Clearly we could assume $\widetilde{\mu}(C) < \infty$. First observe from the definition of $\widetilde{\mu}$ that we can find a descending family $\{A_n\}$ in $\mathcal{M}$ such that $C \subset A_n$ for each n and $\widetilde{\mu}(C) = \widetilde{\mu}(A_\infty)$ where $A_\infty = \bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$. We have

$$\begin{aligned}
\widetilde{\mu}(C) &= \widetilde{\mu}(A_\infty) \\
&= \mu(A_\infty) \\
&= \mu(A_\infty \cap E) + \mu(A_\infty \setminus E) \\
&= \widetilde{\mu}(A_\infty \cap E) + \widetilde{\mu}(A_\infty \setminus E) \\
&\geq \widetilde{\mu}(C \cap E) + \widetilde{\mu}(C \setminus E),
\end{aligned}$$

done.

To show (b), assume that $\mu(X) < \infty$ first. For $E \in \mathcal{M}_C$, there exists some $A \in \mathcal{M}$ such that $E \subset A$ and $\widetilde{\mu}(E) = \mu(A)$. Thus $\widetilde{\mu}(A \setminus E) = \mu(A) - \widetilde{\mu}(E) = 0$. Using the definition of $\widetilde{\mu}$ again, we can find some $N \in \mathcal{M}, A \setminus E \subset N$ satisfying $\mu(N) = \widetilde{\mu}(A \setminus E) = 0$. It shows that $A \setminus E$ is a subset of a set of μ-measure zero. By the completeness of μ, $A \setminus E$ is measurable, so is E as it can be expressed as $A \setminus (A \setminus E)$.

When X is σ-finite, we can find measurable sets X_j with finite measure such that $X = \cup_j X_j$. For $E \in \mathcal{M}_C$, $E \cap X_j$ has finite measure for each j. By the proof above, $E \cap X_j \in \mathcal{M}$, so $E = \cup_j \big(E \cap X_j\big) \in \mathcal{M}$ too. $\square$

2.2 Topological and Metric Spaces

A subset τ of $\mathcal{P}_X$ is called a topology on X if it satisfies the following conditions:

(i) $\phi,\, X \in \tau$,

(ii) $A_\alpha \in \tau \;\Rightarrow\; \cup_\alpha A_\alpha \in \tau$, and

(iii) $A_j \in \tau,\ j = 1, \ldots, N \;\Rightarrow\; \cap_{j=1}^N A_j \in \tau$.

Note that in (ii) the union is taken over any index set, but in (iii) the intersection must be over a finite set. Any element in τ is called an open set and the pair (X, τ) is called a topological space. We recall

- F is called a closed set if its complement F' is open.
- X and ϕ are both open and closed.
- N is a neighborhood of x if there exists an open set G such that $x \in G \subset N$.
- x is an interior point of a set $A \subset X$ if there exists an open set G such that $x \in G \subset A$. Denote all interior points of A by A°. It is the largest open set contained in A. More precisely, A° contains all open subsets of A.
- For $A \subset X$, the closure of A, $\overline{A}$, is the intersection of all closed sets containing A. One can show that $(\overline{A})' = (A')^\circ$. The closure of A is the smallest closed set containing A. In other words, $\overline{A}$ is contained in any closed set containing A. The boundary of A, ∂A, is given by $\overline{A} \cap \overline{A'}$.
- The set A is compact if whenever $A \subset \cup_\alpha G_\alpha$ for a family of open sets $\{G_\alpha\}$, there exist $\alpha_1, \ldots, \alpha_N$ such that $A \subset \cup_{j=1}^N G_{\alpha_j}$.

We now consider metric spaces. A function d from $X \times X \to [0, \infty)$ is called a metric if it satisfies, for $\forall x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a metric space. The metric induces a topology on X as follows. Call a set G open if for each $x \in G$, there exists $B_r(x) \subset G$ where $B_r(x) = \{y : d(y, x) < r\}$ is the metric ball of radius r centered at x $\{y : d(y, x) < r\}$. The collection of all open sets forms a topology on X. For example, the standard topology in $\mathbb{R}^n$ is induced by the Euclidean metric.

Let (X, τ) be a topological space. We use $\mathcal{B}$ to denote the σ-algebra generated by τ. Elements in $\mathcal{B}$ are called *Borel sets.* All open and closed sets are Borel sets. There are more, for instance, a set A is called a G_δ*-set* if $A = \bigcap_{j=1}^\infty G_j$, G_j open and it is called an F_σ*-set* if $A = \bigcup_{j=1}^\infty F_j$, F_j closed. Both G_δ and F_σ sets are Borel sets. For instance, the closed-open interval

$$
\begin{aligned}
[a, b) &= \bigcup_{n=1}^\infty [a, b - \frac{1}{n}] \\
&= \bigcap_{n=1}^\infty (a - \frac{1}{n}, b)
\end{aligned}
$$

is a F_σ- and a G_δ-set in $\mathbb{R}$ at the same time. However, there are G_δ-sets which are not F_σ-sets, and vice versa in $\mathbb{R}$.

Proposition 2.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space where X is a topological space and $\mathcal{B} \subset \mathcal{M}$. Every continuous function $f : X \to \overline{\mathbb{R}}$ is measurable.*

Proof. That f is continuous means $f^{-1}(G)$ is open for all open $G \subset \overline{\mathbb{R}}$. So $f^{-1}(G) \in \mathcal{B} \subset \mathcal{M}$. $\square$

A measure μ on a measure space $(X, \mathcal{M}, \mu)$ is called a *Borel measure* if $\mathcal{M}$ contains all Borel sets. An outer measure μ is called a Borel measure if μ is a Borel measure on $(X, \mathcal{M}_C, \mu)$.

The following Caratheodory's criterion is very useful in verifying Borel measurability when the underlying space is a metric space. It will be used in the next chapter when we discuss Hausdorff measures.

Theorem 2.4 (**Caratheodory's Criterion**). *Let (X, d) be a metric space and μ an outer measure on X satisfying*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever dist$(A, B) \equiv \inf\{d(x, y) : \ x \in A, \ y \in B\} > 0$. Then μ is a Borel measure.

Note that $\mathcal{M}_C$ may be larger than $\mathcal{B}$. We know that this is the case for the Lebesgue measure on $\mathbb{R}$.

Proof. It suffices to show that all closed sets are measurable. Let A be closed. We would like to show

$$\mu(C) \geq \mu(C \cap A) + \mu(C \setminus A), \quad \forall C \subset X.$$

As usual, we may assume $\mu(C) < \infty$. Let

$$A_n = \left\{ x \in X : \quad d(x, A) \leq \frac{1}{n} \right\}.$$

Then $C \cap A$ and $C \setminus A_n$ have positive distance. We have

$$\begin{aligned} \mu(C) &\geq \mu(C \cap A \cup C \setminus A_n) \\ &= \mu(C \cap A) + \mu(C \setminus A_n). \end{aligned}$$

Let

$$R_k = \left\{ x \in X : \quad \frac{1}{k+1} < d(x, A) \leq \frac{1}{k} \right\}.$$

We claim that $A_n = A \cup \cup_{k=n}^{\infty} R_k$. For, if $x \in A_n$ satisfies $d(x, A) = 0$, then $x \in A$ because A is a closed set. On the other hand, if $d(x, A_n) > 0$, there exists some $k \geq n$ such that $x \in R_k$. It follows that $A = A_n \setminus \cup_{k=n}^{\infty} R_k$ and

$$\begin{aligned} C \setminus A &= (C \setminus A_n) \cup \left(C \cap \cup_{k=n}^{\infty} R_k \right) \\ &= (C \setminus A_n) \cup \left(\cup_{k=n}^{\infty} \left(R_k \cap C \right) \right) \end{aligned}$$

holds and therefore,

$$\begin{aligned} \mu(C \setminus A) &\leq \mu(C \setminus A_n) + \mu\left(\cup_{k=n}^{\infty} \left(R_k \cap C \right) \right) \\ &\leq \mu(C \setminus A_n) + \sum_{k=n}^{\infty} \mu(R_k \cap C). \end{aligned}$$

If we can show that

$$\sum_{k=1}^{\infty} \mu(R_k \cap C) < \infty, \tag{2.2}$$

then $\sum_{k=n}^{\infty} \mu(R_k \cap C) \to 0$ as $n \to \infty$. As a result,

$$\begin{aligned} \mu(C \setminus A) &\leq \lim_{n \to \infty} \mu(C \setminus A_n) + \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(R_k \cap C) \\ &\leq \lim_{n \to \infty} \mu(C \setminus A_n), \end{aligned}$$

and

$$\mu(C) \geq \mu(C \cap A) + \lim_{n \to \infty} \mu(C \setminus A_n) \geq \mu(C \cap A) + \mu(C \setminus A),$$

the theorem follows.

To prove (2.2), we split the sum into

$$\sum_{k=1}^{\infty} \mu(R_{2k} \cap C) \text{ and } \quad \sum_{k=1}^{\infty} \mu(R_{2k+1} \cap C).$$

We claim that the distance between R_k and R_{k+2} is at least $1/(k+1)(k+2)$. For, let $y \in R_{k+2}$. For $\varepsilon > 0$, there exists some $z \in A$ such that $d(y,z) \le 1/(k+2)+\varepsilon$. Using $d(x,y) + d(y,z) \ge d(x,z) \ge d(x,A) \ge 1/(k+1)$, we have $d(x,y) + \varepsilon \ge 1/(k+1) - 1/(k+2) = 1/(k+1)(k+2) > 0$, and the claim follows by letting $\varepsilon \to 0$. As the distance between any two of $R_2, R_4, \ldots, R_{2N}$ is positive,

$$\mu(\cup_{k=1}^{N} R_{2k} \cap C) = \sum_{k=1}^{N} \mu(R_{2k} \cap C).$$

Letting $N \to \infty$,

$$\sum_{k=1}^{\infty} \mu(R_{2k} \cap C) = \mu\left(\cup_{k=1}^{\infty} R_{2k} \cap C\right) \le \mu(C) < \infty.$$

and similarly, one can show that

$$\sum_{k=1}^{\infty} \mu(R_{2k+1} \cap C) < \infty.$$

$\square$

2.3 Locally Compact Hausdorff Spaces

Two types of topological spaces are usually employed as the platform for performing integration. The first is metric spaces and the second is locally compact Hausdorff spaces. Both spaces have a large supply of continuous functions.

A topological space is a Hausdorff space if any two distinct points in it can be separated by disjoint open sets, that is, for $x \neq y$ in X there exist open G_1 and G_2 containing x and y respectively such that $G_1 \cap G_2 = \phi$. A topological space is locally compact if each point has a compact neighborhood N. We refer to p.35-40, [R] for further information on topological spaces. For those who do not want to go in topology, you may simply assume the topological space is a Euclidean space $\mathbb{R}^n$.

The following are two fundamental properties of a locally compact Hausdorff space we will use from time to time.

Proposition 2.5. *Let $K \subset G$ where K is compact and G is open in a locally* *compact Hausdorff space. Then there exists an open set V with $\overline{V}$ compact such that $K \subset V \subset \overline{V} \subset G$.*

We will use the notation $V \subset\subset G$ to denote the situation that $\overline{V}$ is compact and contained in G.

Theorem 2.6 (**Urysohn's Lemma**). *Let $K \subset G$ where G is open and K is compact in a locally compact Hausdorff space. Then there exists a continuous function f with compact support in G such that $0 \leq f \leq 1$ in X and $f \equiv 1$ on K.*

Since a single point is a compact set, this proposition implies that for every point x in an open set G, there always exists a continuous function compactly supported in G and satisfies $f(x) = 1$. Proposition 2.5 and Theorem 2.6 are proved in [R]. A simpler proof can be constructed when the space is $\mathbb{R}^n$. In fact, there are several versions of the Urysohn's lemma and you may google for more information.

The *support* of a function f defined to be

$$\mathrm{spt} f \equiv \overline{\{x : f(x) \neq 0\}}.$$

So it is a closed set by definition. Let x and y be two distinct points in X. Taking K to be x and G to be the complement of y, one concludes from the Urysohn's lemma that there is always a continuous function which is equal to 1 at x and 0 at y. This demonstrates that there are many continuous functions in X.

The notation $f < G$ means $f \in C_c(X), 0 \leq f \leq 1$, and $\mathrm{spt} f \subset G$.

The following assertion on the partition of unity is a useful technical tool.

Theorem 2.7. *Let X be a locally compact Hausdorff space and $K \subset \bigcup_{j=1}^N G_j$ where K is compact and G_j's are open. There exist $\varphi_j < G_j, j = 1, \cdots, N$, satisfying $\sum_{j=1}^N \varphi_j \equiv 1$ on K.*

Proof. (Following [R]) For each $x \in K$, there is an open set W_x containing x such that $\overline{W_x}$ is compact, and is contained in one of these G_j. There is a finite covering of K by $W_{x_1}, \ldots, W_{x_m}$. Let H_j be the union of all those $\overline{W_{x_k}}$ contained in G_j. Then H_j is a compact set in G_j and $K \subset \displaystyle\bigcup_{j=1}^N H_j$. By Urysohn's lemma, there exist $f_j \in C(X)$, $0 \leq f_j \leq 1$, $f_j \equiv 1$ on H_j and $\mathrm{spt} f_j \subset G_j$. We set

$$\begin{aligned}
\varphi_1 &= f_1 \\
\varphi_2 &= (1 - f_1) f_2 \\
&\vdots \\
\varphi_N &= (1 - f_1)(1 - f_2) \cdots (1 - f_{N-1}) f_N.
\end{aligned}$$

By a routine induction one can show that

$$\varphi_1 + \varphi_2 + \cdots + \varphi_N = 1 - (1 - f_1)(1 - f_2)\cdots(1 - f_N),$$

so $\sum_1^N \varphi_j \equiv 1$ on K. Clearly, $\varphi_j \in [0,1]$ and spt $\varphi_j \subset G_j$. $\square$

2.4 Riesz Representation Theorem

We denote the set of all continuous functions with compact support in a topological space X by $C_c(X)$. It carries the structure of a vector space. Recall that a linear functional is a linear map from a vector space to $\mathbb{R}$. A linear functional Λ on $C_c(X)$ is called *positive* if

$$\Lambda f \geq 0, \quad \forall f \geq 0 \text{ in } C_c(X).$$

Given a Borel measure μ which is finite on compact sets, we can associate it with a positive functional Λ on $C_c(X)$ given by

$$f \mapsto \int f d\mu.$$

Indeed, let K be the support of f, we have

$$\begin{aligned} \int |f|\ d\mu &= \int_K |f| d\mu \\ &\leq \sup_x |f(x)|\mu(K) \\ &< \infty, \end{aligned}$$

whence $\int |f| d\mu$ is finite. It shows that every function in $C_c(X)$ is integrable. In particular, Λ is finite on $C_c(X)$. It is not at all clear that the construction can be reversed. This is the content of the Riesz representation theorem. In fact, we construct an outer measure from a positive functional as follows. Let G be an open set in the locally compact Hausdorff space X. By Urysohn's lemma there exists at least one continuous function with values in $[0,1]$ compactly supported in G. The set $\{f \in C_c(X) : \ f < G\ \}$ is non-empty and we define

$$\mu_0(G) = \sup\big\{\Lambda f : \ f < G\big\},$$

and $\mu_0(\phi) = 0$. For $E \subset X$, define

$$\mu(E) = \inf\big\{\mu_0(G) : E \subset G,\ G \text{ is open}\big\}.$$

Theorem 2.8 (**Riesz Representation Theorem**). *Let X be a locally compact Hausdorff space and Λ a positive linear functional on $C_c(X)$. Then μ defined* *above is a Borel (outer) measure which is finite on compact sets and*

$$\Lambda f = \int f \, d\mu, \quad \forall f \in C_c(X),$$

holds.

Proof. It is clear from the definition of μ_0 that $\mu_0(G_1) \leq \mu_0(G_2$ for open sets $G_1, G_2, G_1 \subset G_2$. It follows from the definition of μ that μ coincides with μ_0 on open sets. We divide the proof of the representation theorem into four steps.

1. μ is an outer measure.
2. All Borel sets are μ-measurable.
3. μ is finite on compact sets.
4. Identifying Λ with the integral w.r.t. μ.

Step 1. Let $E \subset \bigcup_{j=1}^{\infty} E_j, E_j \subset X, j \geq 1$. We claim

$$\mu(E) \leq \sum_{1}^{\infty} \mu(E_j).$$

We can assume $\sum_1^\infty \mu(E_j) < \infty$, for otherwise there is nothing to prove. For $\varepsilon > 0$, there is some open G_j containing E_j such that

$$\mu(E_j) + \frac{\varepsilon}{2^j} \geq \mu(G_j).$$

The set $G = \bigcup_j G_j$ is open and contains E. Let $f < G$ and consider the compact set $K = \text{spt} f$. There are finitely many G_j's such that $K \subset \bigcup_{j=1}^{N} G_{n_j}$. Let φ_j be a partition of unity subordinate to $\{G_{n_j}\}$. Then $\sum_j \varphi_j = 1$ on K. We have

$$\begin{aligned}
\Lambda f &= \sum_{1}^{N} \Lambda(f\varphi_j) \\
&\leq \sum_{j=1}^{N} \mu(G_{n_j}) \quad (\text{since } f\varphi_j < G_{n_j}) \\
&\leq \sum_{j=1}^{\infty} \mu(G_j) \\
&\leq \sum_{j=1}^{\infty} \mu(E_j) + \varepsilon.
\end{aligned}$$

Taking supremum over all these f's, we have

$$\begin{aligned}\mu(E) &\leq \mu(G) \\ &= \sup\{\Lambda f : f < G\} \\ &\leq \sum_{j=1}^{\infty} \mu(E_j) + \varepsilon,\end{aligned}$$

and claim follows by letting $\varepsilon \to 0$.
Step 2. It suffices to show for every open E,

$$\mu(C) \geq \mu(C \cap E) + \mu(C \setminus E), \quad \forall C \subset X.$$

From the definition of μ, it suffices to show, for every open E,

$$\mu(G) \geq \mu(G \cap E) + \mu(G \setminus E), \quad \forall \text{ open } G. \tag{2.3}$$

For $\varepsilon > 0$, pick $\varphi < G \cap E$ such that

$$\Lambda\varphi + \varepsilon \geq \mu(G \cap E).$$

Let $K \equiv \text{spt}\varphi \subset G \cap E$ and $\psi < G \setminus K$. Then $\varphi + \psi < G$ and

$$\begin{aligned}\mu(G) &\geq \Lambda(\varphi + \psi) = \Lambda\varphi + \Lambda\psi \\ &\geq \mu(G \cap E) - \varepsilon + \Lambda\psi.\end{aligned}$$

Taking supremum over all ψ,

$$\begin{aligned}\mu(G) &\geq \mu(G \cap E) - \varepsilon + \mu(G \setminus K) \\ &\geq \mu(G \cap E) - \varepsilon + \mu(G \setminus E)\end{aligned}$$

and (2.3) holds after letting $\varepsilon \to 0$.

Step 3. We show that for every compact set K ,

$$\mu(K) = \inf\{\Lambda f : K < f\}.$$

In particular, it implies that μ is finite on compact sets. Here $\{K < f\}$ means $f \in C_c(X), 0 \leq f \leq 1$ and $f \equiv 1$ on K. Such functions exist by Urysohn's lemma. For $\alpha \in (0, 1)$, $G_\alpha = \{x : f(x) > \alpha\}$ is open and $K \subset G_\alpha$. For any $\varphi < G_\alpha$, we have $\varphi \leq f/\alpha$. As $\mu(G_\alpha) = \sup \Lambda\varphi$,

$$\mu(K) \leq \mu(G_\alpha) \leq \frac{1}{\alpha}\Lambda f.$$

Letting $\alpha \uparrow 1$, we obtain $\mu(K) \leq \Lambda f$. By taking infimum over all f, $f < K$, we have

$$\mu(K) \leq \inf \{\Lambda f : K < f\}.$$

On the other hand, for $\varepsilon > 0$, we can find an open set G containing K such that

$$\mu(K) + \varepsilon \geq \mu(G).$$

By Urysohn's lemma, there exists some $\varphi_1,\ K < \varphi_1 < G$. We have

$$\mu(K) + \varepsilon \geq \mu(G) \geq \Lambda\varphi_1,$$

which implies

$$\mu(K) + \varepsilon \geq \inf \{\Lambda\varphi : K < \varphi\},$$

and

$$\mu(K) \geq \inf \{\Lambda\varphi : K < \varphi\},$$

after letting $\varepsilon \downarrow 0$.
Step 4. (Following [R]) It suffices to show

$$\Lambda f \leq \int f \, d\mu, \quad \forall f \in C_c(X).$$

For, if this is true, the reverse inequality follows from replacing f by $-f$ in this inequality. Let $f(X) \in [a, b]$. For $\varepsilon > 0$, pick

$$y_0 < a < y_1 < \cdots < y_n = b,$$

so that $y_j - y_{j-1} < \varepsilon$. Let $E_j = f^{-1}((y_{j-1}, y_j]) \cap K,\ K \equiv \mathrm{spt} f$. Then $\{E_j\}$ are measurable, mutually disjoint and $K = \bigcup_{j=1}^n E_j$. As $\mu(K) < \infty$ by Step 3, $\mu(E_j) < \infty,\ \forall j$. We can find an open set G_j containing E_j such that

$$\mu(E_j) + \frac{\varepsilon}{n} \geq \mu(G_j).$$

We could further require G_j to satisfy

$$y_{j-1} - \varepsilon < f(x) < y_j + \varepsilon, \quad \forall x \in G_j.$$

As $K \subset \bigcup_{j=1}^n G_j$, we fix a partition of unity $\{\varphi_j\}$ on K subordinate to $\{G_j\}$.

Then $f = \sum_j f\varphi_j$ in X and

$$\begin{aligned}
\Lambda f &= \sum \Lambda(f\varphi_j) \\
&\le \sum (y_j + \varepsilon)\Lambda\varphi_j = \sum (|a| + y_j + \varepsilon)\Lambda\varphi_j - |a| \sum \Lambda\varphi_j \\
&\le \sum (|a| + y_j + \varepsilon)\mu(G_j) - |a| \sum \Lambda\varphi_j \\
&\quad (|a| + y_j + \varepsilon \ge 0) \\
&\le \sum (y_{j-1} + |a| + 2\varepsilon)\Big(\mu(E_j) + \frac{\varepsilon}{n}\Big) - |a| \sum \Lambda\varphi_j \\
&= \frac{\varepsilon}{n} \sum (y_{j-1} + |a| + 2\varepsilon) + \sum y_{j-1}\mu(E_j) + \sum (|a| + 2\varepsilon)\mu(E_j) \\
&\quad - |a| \sum \Lambda\varphi_j \\
&\le \varepsilon(|a| + b + 2\varepsilon) + \sum y_{j-1}\mu(E_j) \\
&\quad + 2\varepsilon\mu(K) + |a| \Big(\sum \mu(E_j) - \sum \Lambda\varphi_j \Big) \\
&\le \varepsilon(|a| + b + 2\varepsilon) + \sum y_{j-1}\mu(E_j) + 2\varepsilon\mu(K) \\
&\quad + |a| \Big(\mu(K) - \Lambda\big(\sum \varphi_j\big) \Big) \\
&\le \varepsilon(|a| + b + 2\varepsilon) + \sum y_{j-1}\mu(E_j) + 2\varepsilon\mu(K),
\end{aligned}$$

where $\mu(K) - \Lambda\Big(\sum \varphi_j\Big) \le 0$ has been used in the last step. On the other hand,

$$\int f\, d\mu = \sum_j \int_{E_j} f\, d\mu \ge \sum_j y_{j-1}\mu(E_j).$$

Hence

$$\Lambda f \le \varepsilon(|a| + b + 2\varepsilon) + \int f\, d\mu + 2\varepsilon\mu(K),$$

and the desired result follows after letting $\varepsilon \to 0$.

$\square$

The measure μ may be written as μ_Λ to emphasize its dependence on Λ. We may call it the "Riesz measure" associated to Λ.

The Riesz measure has further regularity properties. Here regularity means how close we can approximate a set by open sets from outside or by compact sets from inside. To state them we need to introduce some terminologies.

Let X be a topological space and μ a Borel measure on it. A set $E \subset X$ is called *outer regular* (w.r.t. μ) if for every $\varepsilon > 0$, there exists some open set $G, E \subset G$, such that $\mu(G) \le \mu(E) + \varepsilon$. It is *inner regular* (w.r.t. μ) if for every $\varepsilon > 0$, there exists a compact $K \subset E$ such that $\mu(E) \le \mu(K) + \varepsilon$.

Proposition 2.9. *Let μ_Λ be the Riesz measure of Λ. Under the setting of Theorem 2.7,*

(a) Every set is outer regular w.r.t. μ_Λ.

(b) Every open set is inner regular w.r.t. μ_Λ.

(c) Every measurable set with finite measure is inner regular w.r.t. μ_Λ.

Proof. (a) It follows immediately from the definition of the Riesz measure.
(b) We have

$$\begin{aligned}\mu(G) &= \sup\{\Lambda f : f < G\} \\ &= \sup\left\{\int f\, d\mu : f < G\right\} \qquad \text{(Theorem 2.7)} \\ &\le \sup\{\mu(K) : \ K \subset G \text{ is compact }\}\end{aligned}$$

since

$$\int f\, d\mu \le \mu(K) \text{ where } K \equiv \text{spt} f \subset G.$$

(c) Let A be measurable, $\mu(A) < \infty$. Given $\varepsilon > 0$, we find an open $G \supset A$ such that $\mu(G) \le \mu(A) + \varepsilon$. By countable additivity, $\mu(G \setminus A) < \varepsilon$. Using (a), there is some open G_1 contained in G such that $G \setminus A \subset G_1$ with $\mu(G_1) < 2\varepsilon$. Then $G \setminus G_1 \subset A$. Noting $A \subset G \setminus G_1 \cup (A \cap G_1)$, we have $\mu(A) \le \mu(G \setminus G_1) + \mu(A \cap G_1) \le \mu(G \setminus G_1) + 2\varepsilon$, so $\mu(G \setminus G_1) + 2\varepsilon \ge \mu(A)$. As G is open, by (b) there exists compact $K \subset G$ such that $\mu(G \setminus K) < \varepsilon$. Then $K_1 \equiv K \setminus G_1$ is compact, $K_1 \subset G \setminus G_1 \subset A$. Writing $G \setminus G_1 = K_1 \cup (G \setminus K) \setminus G_1$, we have

$$\mu(K_1) + 3\varepsilon \ge \mu(A).$$

□

Examples (see Exercise) show that some measurable sets of infinite measure are not inner regular. To remedy this undesirable situation we need to put extra assumption on the space or the measure. Recall that a measure on a topological space X is σ-finite if there exist measurable $\{X_j\}$ such that $X = \bigcup_j X_j$. Without loss of generality one may assume that X_j's are mutually disjoint.

Proposition 2.10. *Let μ be a Riesz measure on a locally compact Hausdorff space X which is σ-finite with respect to μ. Then*

(a) For every measurable set E and given $\varepsilon > 0$, there exist an open set G and a closed F such that $F \subset E \subset G$ and $\mu(G \setminus F) < \varepsilon$.

(b) For each measurable E, there exist a G_δ set A and an F_σ set B such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Consequently, $\mathcal{M}_C$ is the completion of $\mathcal{B}$.

(c) Every measurable set is inner regular.

Proof. (a) Let E be measurable and $E_j = E \cap X_j$ where $X = \bigcup_j X_j$ in a σ-finite decomposition of X. By outer regularity of the Riesz measure, for each $\varepsilon > 0$, there exists an open set G_j containing E_j such that $\mu(G_j \setminus E_j) = \mu(G_j) - \mu(E_j) \leq \varepsilon/2^j$ for all $j \geq 1$. It follows that $\mu(G \setminus E) \leq \sum_j \mu(G_j \setminus E_j) \leq \varepsilon$ where $G = \bigcup_j G_j$ is open after using $G \setminus E = (\bigcup_j G_j) \setminus (\bigcup_k E_k) = \bigcup_j (G_j \setminus \bigcup_k E_k) \subset \bigcup_j (G_j \setminus E_j)$. Next, we apply this result to the complement of E, E', to get an open G_1 such that $E' \subset G_1$ and $\mu(G_1 \setminus E') < \varepsilon$. Then the closed set $F = G'$ is contained in E and satisfies $\mu(E \setminus F) = \mu(G_1 \setminus E') < \varepsilon$.

(b) For each j, there exists an open G_j containing E such that $\mu(G_j \setminus E) < 1/j$. Letting $A = \bigcap_j G_j$, A is a G_δ set containing E and satisfies $\mu(A \setminus E) \leq \mu(G_j \setminus E) < 1/j$ for all j. Letting $j \to \infty$, we conclude that $\mu(A \setminus E) = 0$. Applying this result to E' instead of E, we obtain a G_δ set A_1 containing E' with $\mu(A_1 \setminus E') = 0$. Therefore, the F_σ-set $B = A_1'$ is contained in E and $\mu(E \setminus B) = \mu(A_1 \setminus E') = 0$ holds.

(c) In view of Proposition 2.9(c), it suffices to show that

$$\sup \left\{ \mu(K) : \quad K \subset E \text{ is compact } \right\} = \infty$$

when E is a measurable set with infinite measure. Assuming X_j's are mutually disjoint in a σ-finite decomposition of X and $E_j = E \cap X_j$ as above, we have $\infty = \mu(E) = \sum_j \mu(E_j)$ and therefore for each $M > 0$, we can find some J such that $\sum_{j=1}^{J} \mu(E_j) \geq M - 1$. For each j, let K_j be a compact set in E_j such that $\mu(K_j) \geq \mu(E_j) - 1/2^j$. Then the compact set $K = \cup_{j=1}^{J} K_j$ is contained in E and $\mu(K) = \sum_{j=1}^{J} \mu(K_j) \geq M$, and the desired result holds.

□

If one prefers to impose assumptions on the space itself rather than the measure, one may use the notion of σ-compactness. Indeed, a topological space is *σ-compact* if it can be written as a countable union of compact subsets. The Euclidean space is an example of a σ-compact space. Apparently, every Riesz measure on such a space is σ-finite.

Finally, we have the following rather simple characterization of Riesz measures in $\mathbb{R}^n$.

Proposition 2.11. *Let λ be a Borel measure on $\mathbb{R}^n$ which is finite on compact sets. There exists a Riesz measure μ such that λ and μ coincide on $\mathcal{B}$.*

I leave the proof as an exercise.

2.5 Lusin's Theorem

Here we study how to approximate measurable functions by continuous functions. This is possible only if the measure has some regularity properties. The following Lusin's theorem is a typical example. You may consult [EG] or google for more.

Theorem 2.12. *Let μ be a Riesz measure on a locally compact Hausdorff space X and let f be a real-valued measurable function vanishing outside some measurable set A, $\mu(A) < \infty$. For every $\varepsilon > 0$, there exists some $g \in C_c(X)$, such that*

$$\mu\big(\{x : \ f(x) \neq g(x)\}\big) < \varepsilon.$$

Moreover, when f is bounded, g can be chosen to satisfy

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|.$$

Proof. (Following [R]) First assume that $0 \leq f < 1$ and A is compact. Recalling in the proof of Theorem 1.6, there is a sequence of simple functions $\{s_j\}$ converging to f in a monotone way. Letting $t_j = s_j - s_{j-1}$ and $t_1 = s_1$, we can write

$$f(x) = \sum_{j=1}^{\infty} t_j(x), \quad \forall x \in X.$$

One can verify that $2^j t_j$ is the characteristic function of some T_n in A. Let G be an open set with compact closure containing A. By the regularity properties of the Riesz measure, we can fix compact sets K_j and open sets G_j satisfying $K_j \subset T_j \subset G_j \subset G$ and $\mu(G_j \setminus K_j) \leq \varepsilon/2^j$. By Urysohn's lemma, we can find $h_j, \ K_j < h_j < G_j$ and so define

$$g(x) = \sum_{j=1}^{\infty} \frac{h_j(x)}{2^j}.$$

By Weierstrass M-test, the series defining g is uniformly convergent, hence g is continuous. It is clear that the support of g is inside G. Moreover, as h_j is equal to $2^j t_j$ in K_j, the set g and f differ has measure less than $\sum_j \mu(G_j \setminus K_j) \leq \sum_j \varepsilon/2^j \leq \varepsilon$.

By an approximation argument, one can remove the compactness of A and the boundedness of f as long as it is finite a.e..

Finally, when f is bounded, we set $\varphi(z)$ to be z for $z \in [-M, M]$ where $M = \sup_X |f|, M$ and $-M$ respectively for $z > M$ and $z < M$. Then the composite function $g_1(x) = \varphi(g(x))$ meets our additional requirement. □

Corollary 2.13. *Setting as above, let f be a measurable function satisfying $|f| \leq 1$. Then there exists a sequence of functions $\{g_j\}$ in $C_c(X), |g_j| \leq 1$, such that $\lim_{j\to\infty} g_j(x) = f(x)$ a.e..*

Proof. Applying Lusin's theorem to get $g_j \in C_c(X)$ which is equal to f except in a measurable set E_j with $\mu(E_j) \le 2^{-j}$. Then $\sum_j \mu(E_j) \le 1$. By Borel-Cantelli lemma, the set E consisting of points which belong to infinitely many E_j's is a null set. For each x not in this set, it could only belong to finitely many E_j's. Letting the largest j be j_0, $g_j(x) = f(x)$ for all $j > j_0$. $\square$

2.6 Semicontinuous and Borel Functions

Let X be a topological space and $x_0 \in X$. The function $f : X \to (-\infty, \infty]$ is *lower semicontinuous at* x_0 if for every $\varepsilon > 0$, there is a neighborhood N of x_0 such that

$$f(x) > f(x_0) - \varepsilon, \quad \forall x \in N.$$

When $f(x_0) = \infty$, the definition should be modified to: For every $M > 0$, there exists a neighborhood N of x_0 such that

$$f(x) > M, \quad \forall x \in N.$$

We define *upper semicontinuity at* x_0 for $f : X \to [-\infty, \infty)$ as requiring $-f$ to be lower semicontinuous at x_0. A function is lower semicontinuous on a set E, $E \subset X$, if it is so at every point of E.

Proposition 2.14. *A function $f : X \to (-\infty, \infty]$ is lower semicontinuous everywhere if and only if $\{x \in X : f(x) > a\}$ is open for every $a \in \mathbb{R}$.*

Proof. $\Rightarrow$) Let $x_0 \in \{x \in X : f(x) > a\}$. Then $f(x_0) > a$. For $\varepsilon = (f(x_0) - a)/2$, there exists a neighborhood N of x_0 such that

$$\begin{aligned} f(x) &\ge f(x_0) - \frac{1}{2}(f(x_0) - a) \\ &= \frac{1}{2} f(x_0) + \frac{a}{2} \\ &> a, \quad \forall x \in N, \end{aligned}$$

hence $N \subset \{x \in X : f(x) > a\}$ and $\{x \in X : f(x) > a\}$ is an open set.
$\Leftarrow$) Taking $a = f(x_0) - \varepsilon$, then $\{x : f(x) > f(x_0) - \varepsilon\}$ is an open set and contains x_0. $\square$

Note the following elementary facts:

- If f is lower semicontinuous at x_0, then

$$\varliminf_{x \to x_0} f(x) \ge f(x_0).$$

If f is upper semicontinuous at x_0, then

$$\overline{\lim_{x \to x_0}} f(x) \le f(x_0).$$

- Every continuous function is lower and upper semicontinuous.
- The characteristic function χ_G is lower semicontinuous when G is open and χ_F is upper semicontinuous when F is closed.
- The supremum $f = \sup_\alpha f_\alpha$ where f_α's are lower semicontinuous is lower semicontinuous. The infimum $f = \inf_\alpha f_\alpha$ where f_α's are upper semicontinuous is upper semicontinuous.

See Exercise 5 for more.

One can use lower and upper semicontinuous functions to approximate measurable functions. The following Vitali-Carathéodory theorem is a sample.

Theorem 2.15. *Let μ be a Borel outer measure in which all measurable sets with finite measure are regular. For each integrable function f and $\varepsilon > 0$, there exist an upper continuous u which is bounded above and a lower semicontinuous v which is bounded from below such that $u \le f \le v$ and*

$$\int (v - u)\, d\mu < \varepsilon.$$

See [R] for a proof.

An extended real-valued function f is a *Borel function* if $f^{-1}(G)$ is a Borel set for every open set G in $\overline{\mathbb{R}}$. Lower and upper semicontinuous functions constitute a special class of Borel measurable functions. For a Borel measure, we have the following inclusion relations

$$\{\text{continuous functions}\} \subset \{\text{lower/upper semicontinuous functions}\}$$

$$\subset \{\text{Borel functions}\} \subset \{\text{measurable functions}\}.$$

We now study how to approximate a measurable function by Borel functions. When the measure is a Riesz measure, a measurable function is in fact equal to a Borel function modulo a null set. To see this, we start with a general observation.

Lemma 2.16. *Let $(X, \mathcal{M}, \mu)$ and $(X, \mathcal{M}_1, \mu_1)$ be two measure spaces where $\mathcal{M}_1 \subset \mathcal{M}$ and μ coincides with μ_1 on $\mathcal{M}_1$. Suppose that for each $A \in \mathcal{M}$, there exists some $A_1 \in \mathcal{M}_1$ such that $A \subset A_1$ and $\mu(A_1 \setminus A) = 0$. Given any extended real-valued, $\mathcal{M}$-measurable function f from X to $\overline{\mathbb{R}}$, there exists an $\mathcal{M}_1$-measurable g such that $g = f$ μ-a.e.*

Proof. By Theorem 1.6 we can find an increasing sequence of simple functions $\{s_k\}$ satisfying $\lim_{k\to\infty} s_k(x) = f(x)$ for every $x \in X$. Each s_k is of the form

$\sum_j \alpha_j^k \chi_{E_j^k}$ where $E_j^k \in \mathcal{M}$ are mutually disjoint for each fixed k. By assumption, we can find $F_j^k \in \mathcal{M}_1$ such that $E_j^k \subset F_j^k$ and $\mu(F_j^k \setminus E_j^k) = 0$. We set $t_k = \sum_j \alpha_j^k \chi_{F_j^k}$ and $g(x) = \underline{\lim}_{k\to\infty} t_k(x)$. As each t_k is $\mathcal{M}_1$-measurable, so is g. Furthermore, each t_k is equal to s_k μ-a.e.. It follows that g is equal to f μ-a.e.. $\square$

Proposition 2.17. *Let μ be a Borel measure on a topological space X in which every measurable set is outer regular. Suppose that μ is σ-finite. Then for every measurable function from X to $\overline{\mathbb{R}}$, there exists a Borel function g which coincides with f μ-a.e..*

Proof. It suffices to prove the proposition assuming that $\mu(X) < \infty$. By outer regularity, for each measurable set E and n, there exists a Borel set B_n satisfying $E \subset B_n$ and $\mu(B_n \setminus E) = \mu(B_n) - \mu(E) < 1/n$. It follows that the Borel set $B = \bigcap_n B_n$ contains E and $\mu(B) = \mu(E)$. Now we can apply the lemma above to draw the desired conclusion.

$\square$

Comments on Chapter 2. In this chapter we discuss outer measures following [EG] and use it to prove the Riesz representation theorem despite many details are from [R]. A full version of Riesz representation theorem which associated a bounded linear functional on $C_c(X)$ to a signed measure will be discussed in Chapter 5.

The modern theory of integration was due to Lebesgue (1902) who defined the Lebesgue integration on the Euclidean space using basically the outer measure approach. Instead of using closed rectangles to cover a set, he used open rectangles, but this is not essential. On the other hand, Riesz (1909) established the unexpected result that every positive linear functional on $[0,1]$ is a Riemann-Stieltjes integral. Nowadays it is known that each Riemann-Stieltjes integral corresponds to the integral with respect to some Borel measure. Lebesgue's approach was generalized by Caratheodory (1918) in his construction of outer measures. Since then these have been the two main ways to construct measure spaces. In [R] the approach by Riesz representation theorem is preferred and outer measures are not mentioned at all. However, in the past several decades, the advances in analysis (PDE's and dynamical systems) and geometry (fractals and geometric measure theory) have shown the importance of Caratheodory's approach over the approach by the representation theorem. One reason is the restriction of the latter: The measures obtained by the representation are finite on compact sets. However, there are many interesting measures, for instance, fractional Hausdorff measures, are not finite on compact sets and consequently cannot be associated to any positive linear functionals. To make our discussion updated and balanced, in this chapter we discuss outer measures first and use it to prove Riesz representation theorem. Nevertheless, you should not be left with the impression that I am downplaying the role of Riesz representation theorem in real analysis. In fact, both Caratheodory's construction and Riesz representation theorem will be used in our subsequent development.