Solution 1

Section 6.1

5. (a) Since $1+x^2 \neq 0$, $f'(x) = \dfrac{(1+x^2)-(2x)x}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$.

   (b) Since $5-2x+x^2 = 4+(1-x)^2 > 0$, $g'(x) = \dfrac{1}{2}(5-2x+x^2)^{-1/2}(2x-2) = \dfrac{x-1}{\sqrt{5-2x+x^2}}$.

   (c) By chain rule, $h'(x) = m(\sin x^k)^{m-1}(\cos x^k)(kx^{k-1}) = mkx^{k-1}\sin^{m-1} x^k \cos x^k$.

   (d) For $|x| < \sqrt{\pi/2}, x^2 < \pi/2, \tan y$ is differentiable for $y \in (-\pi/2, \pi/2)$.
   By chain rule, we have $k'(x) = (\sec^2 x^2)(2x) = 2x\sec^2 x^2$.

6. Clearly $f'(x) = nx^{n-1}$ for $x>0$ and $f'(x) = 0$ for $x<0$.
   Now for $x>0$, $\dfrac{f(x)-f(0)}{x-0} = \dfrac{x^n-0}{x-0} = x^{n-1} \quad\Rightarrow\quad f'_+(0) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n\geq 2. \end{cases}$
   For $x<0$, $\dfrac{f(x)-f(0)}{x-0} = \dfrac{0-0}{x-0} = 0 \quad\Rightarrow\quad f'_-(0) = 0$
   Now $f'(0)$ exists iff $f'_+(0) = f'_-(0) = 0 \quad\Leftrightarrow\quad n\geq 2$.
   Hence for $n>1$, $f'(x) = \begin{cases} nx^{n-1}, & \text{for } x>0 \\ 0, & \text{for } x\leq 0. \end{cases}$
   Claim: f' is continuous at 0 iff $n\geq 2$. Assume $n=1$, f' is not defined at 0. Conversely, for $n>1$, $\lim_{x\to 0^+} f'(x) = 0 = \lim_{x\to 0^-} f'(x) = f'(0)$. This proves the claim.
   Set $n>1$. For $x>0$, $\dfrac{f'(x)-f'(0)}{x-0} = \dfrac{nx^{n-1}-0}{x-0} = nx^{n-2} \quad\Rightarrow\quad f''_+(0) = \begin{cases} 1, & \text{if } n=2 \\ 0, & \text{if } n\geq 3. \end{cases}$
   For $x<0$, $\dfrac{f'(x)-f'(0)}{x-0} = \dfrac{0-0}{x-0} = 0 \quad\Rightarrow\quad f''_-(0) = 0$.
   Thus f' is differentiable at 0 iff $f''_+(0) = f''_-(0) = 0 \quad\Leftrightarrow\quad n\geq 3$.

7. $\dfrac{g(x)-g(c)}{x-c} = \dfrac{|f(x)|-|f(c)|}{x-c} = \operatorname{sgn}(x-c)\left|\dfrac{f(x)-f(c)}{x-c}\right|$, since $f(c)=0$.
   $g'_+(c) = \lim_{x\to c^+} \operatorname{sgn}(x-c)\left|\dfrac{f(x)-f(c)}{x-c}\right| = |f'(c)|$.
   $g'_-(c) = \lim_{x\to c^-} \operatorname{sgn}(x-c)\left|\dfrac{f(x)-f(c)}{x-c}\right| = -|f'(c)|$.
   Hence g is differentiable at c iff $g'_+(c) = g'_-(c) \quad\Leftrightarrow\quad |f'(c)| = -|f'(c)| \quad\Leftrightarrow\quad f'(c) = 0$.

8. (a) $f(x) = |x|+|x+1| = \begin{cases} 2x+1, & \text{for } x\geq 0 \\ 1, & \text{for } -1\leq x<0 \\ -2x-1, & \text{for } x<-1 \end{cases}$
   Clearly, $f'(x) = \begin{cases} 2, & \text{for } x>0 \\ 0, & \text{for } -1<x<0 \\ -2, & \text{for } x<-1 \end{cases}$
   For $x>0$, $\dfrac{f(x)-f(0)}{x-0} = \dfrac{(2x+1)-1}{x-0} = 2 \quad\Rightarrow\quad f'_+(0) = 2$
   For $x<0$, $\dfrac{f(x)-f(0)}{x-0} = \dfrac{1-1}{x-0} = 0 \quad\Rightarrow\quad f'_-(0) = 0 \neq 2 = f'_+(0)$.
   Similar procedures proceed for $x<-1, x>-1$.
   Hence f is differentiable except $0, -1$.

(b) $g(x) = 2x + |x| = \begin{cases} 3x, & \text{for } x \geq 0 \\ x, & \text{for } x < 0 \end{cases}$

Clearly, $g'(x) = \begin{cases} 3, & \text{for } x > 0 \\ 1, & \text{for } x < 0 \end{cases}$

For $x > 0$, $\dfrac{g(x) - g(0)}{x - 0} = \dfrac{3x - 0}{x - 0} = 3 \quad \Rightarrow \quad g'_+(0) = 3$

For $x < 0$, $\dfrac{g(x) - g(0)}{x - 0} = \dfrac{x - 1}{x - 0} = 1 \quad \Rightarrow \quad g'_-(0) = 1.$

Hence g is differentiable except 0.

(c) $h(x) = x|x| = \begin{cases} x^2, & \text{for } x \geq 0 \\ -x^2, & \text{for } x < 0 \end{cases}$

Clearly, $h'(x) = \begin{cases} 2x, & \text{for } x > 0 \\ -2x, & \text{for } x < 0 \end{cases}$

For $x > 0$, $\dfrac{h(x) - h(0)}{x - 0} = \dfrac{x^2 - 0}{x - 0} = x \quad \Rightarrow \quad h'_+(0) = 0$

For $x < 0$, $\dfrac{h(x) - h(0)}{x - 0} = \dfrac{-x^2 - 0}{x - 0} = -x \quad \Rightarrow \quad h'_-(0) = 0.$

Hence h is differentiable on the whole $\mathbb{R}$.

(d) $k(x) = |\sin x| = \begin{cases} \sin x, & \text{for } \sin x \geq 0 \quad \Leftrightarrow \quad 2n\pi \leq x \leq (2n+1)\pi \\ -\sin x, & \text{for } \sin x < 0 \quad \Leftrightarrow \quad (2n-1)\pi < x < 2n\pi \end{cases}, \forall n \in \mathbb{Z}.$

Clearly, $k'(x) = \begin{cases} \cos x, & \text{for } 2n\pi < x < (2n+1)\pi \\ -\cos x, & \text{for } (2n-1)\pi < x < 2n\pi \end{cases}, \forall n \in \mathbb{Z}.$

For $n \in \mathbb{Z}$ and $x > 2n\pi$, $\dfrac{k(x) - k(2n\pi)}{x - 2n\pi} = \dfrac{\sin x}{x - 2n\pi} = \dfrac{\sin(x - 2n\pi)}{x - 2n\pi} \quad \Rightarrow \quad k'_+(2n\pi) = 1$

For $n \in \mathbb{Z}$ and $x < 2n\pi$, $\dfrac{k(x) - k(2n\pi)}{x - 2n\pi} = \dfrac{-\sin x}{x - 2n\pi} = -\dfrac{\sin(x - 2n\pi)}{x - 2n\pi}$

$\Rightarrow \quad k'_-(2n\pi) = -1$

Similar procedures proceed for $x < (2n+1)\pi, x > (2n+1)\pi, n \in \mathbb{Z}$.

Hence, k is differentiable except $n\pi$ for $n \in \mathbb{Z}$.

9. $f'(-x) = \dfrac{f(-x+h) - f(-x)}{h} = -\lim_{h \to 0} \dfrac{f(x-h) - f(x)}{-h} = -\lim_{h' \to 0} \dfrac{f(x+h') - f(x)}{h'} = -f'(x).$

Hence f' is an odd function.

$g'(-x) = \dfrac{g(-x+h) - g(-x)}{h} = \lim_{h \to 0} \dfrac{[-g(x-h)] - [-g(x)]}{-(-h)} = \lim_{h' \to 0} \dfrac{g(x+h') - g(x)}{h'} = g'(x).$

Hence g' is an even function.

10. For $x \neq 0$, $g'(x) = 2x\ \sin\dfrac{1}{x^2} + x^2 \cos\dfrac{1}{x^2}(-2x^{-3}) = 2x\sin\dfrac{1}{x^2} - \dfrac{2}{x}\cos\dfrac{1}{x^2}.$

$\left|\dfrac{g(x) - g(0)}{x - 0}\right| = \left|\dfrac{x^2\ \sin\frac{1}{x^2} - 0}{x - 0}\right| = \left|x\sin\dfrac{1}{x^2}\right| \leq |x| \quad \Rightarrow \quad g'(0) = \lim_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = 0$

Hence g is differentiable at all $x \in \mathbb{R}$.

Next we show g' is unbounded. Take $x_n := 1/\sqrt{2n\pi}\ \forall\ n \in \mathbb{N}$.

$g'(x_n) = \dfrac{2}{\sqrt{2n\pi}}\sin(2n\pi) - 2\sqrt{2n\pi}\cos(2n\pi) = -2\sqrt{2n\pi}$ is unbounded.

Hence result follows.

13. Denote $g(h) := \dfrac{f(c+h) - f(c)}{h}$. Hence $\lim\limits_{h\to 0} g(h) = \lim\limits_{h\to 0} \dfrac{f(c+h) - f(c)}{h} = f'(c) \in \mathbb{R}$.
By sequential criterion for limits (Theorem 4.1.8 page 101), denote $h_n := 1/n \neq 0$ for all n, and $\lim h_n = \lim \dfrac{1}{n} = 0$, we have $\lim g(h_n) = \lim\limits_{h\to 0} g(h) = f'(c)$, where
$g(h_n) = \dfrac{f(c+1/n) - f(c)}{1/n} = n\{f(c+1/n) - f(c)\}$. Hence $f'(c) = \lim\left(n\{f(c+1/n) - f(c)\}\right)$.
Take $f(x) := \begin{cases} \sin \pi/x, & x > 0 \\ 0, & x \leq 0. \end{cases}$
At $c = 0$, $n\{f(1/n) - f(0)\} = n(0 - 0) = 0 \ \forall\ n$.
Hence, $\lim\left(n\{f(c+1/n) - f(c)\}\right) = 0$.
However, $f'(c)$ doesn't exist because f is not continuous at c.

Or, we may take $f := \chi_{\mathbb{Q}}$ = Dirichlet function. Fix $c \in \mathbb{R}$.
Then $n\{f(c+1/n) - f(c)\} = \begin{cases} n(1-1), & c \in \mathbb{Q} \\ n(0-0), & c \notin \mathbb{Q} \end{cases} = 0 \ \forall\ n$.
The Dirichlet function $\chi_{\mathbb{Q}}$ is not continuous.

Remark If x is rational and y is irrational, why is $x + y$ irrational?

14. Now $h'(x) = 3x^2 + 2 > 0 \ \forall\ x \in \mathbb{R}$. Hence, by Theorem 6.1.8, h^{-1} is differentiable and $(h^{-1})'(y) = \dfrac{1}{h'(x)} = \dfrac{1}{3x^2+2} \quad \forall\ x \in \mathbb{R}$,
where y is related to x by $y = h(x)$.
For $x = 0$, we have $y = h(0) = 1$, and $(h^{-1})'(1) = \dfrac{1}{3(0)+2} = \dfrac{1}{2}$
For $x = 1$, we have $y = h(1) = 4$, and $(h^{-1})'(4) = \dfrac{1}{3(1)+2} = \dfrac{1}{5}$
For $x = -1$, we have $y = h(-1) = -2$, and $(h^{-1})'(-1) = \dfrac{1}{3(1)+2} = \dfrac{1}{5}$.

Section 6.2

19. Let $\varepsilon > 0$. By uniform differentiability, $\exists\ \delta := \delta(\varepsilon) > 0$ s.t. if $0 < |x - y| < \delta$, then

$$\left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| < \frac{\varepsilon}{2}, \quad \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \frac{\varepsilon}{2}$$

It follows that

$$|f'(x) - f'(y)| \leq \left|\frac{f(x) - f(y)}{x - y} - f'(x)\right| + \left|\frac{f(x) - f(y)}{x - y} - f'(y)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence f' is continuous on I.

Supplementary Exercise

1. $\dfrac{f(c+h)-f(c-h)}{2h}$
$= \dfrac{f(c+h)-f(c)+f(c)-f(c-h)}{2h} = \dfrac{1}{2}\dfrac{f(c+h)-f(c)}{h} + \dfrac{1}{2}\dfrac{f(c-h)-f(c)}{-h}$
Hence we have

$$\begin{aligned} f'_s(c) &= \lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h} \\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h} \\ &= \frac{1}{2}\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} + \frac{1}{2}\lim_{h'\to 0}\frac{f(c+h')-f(c)}{h'} \\ &= \frac{1}{2}f'(c) + \frac{1}{2}f'(c) = f'(c) \end{aligned}$$

Observation The set-up for $f'_s(c) = \lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}$ doesn't involve the value $f(c)$, a simple idea to construct a counter example is by changing the value $f(c)$ from a differentiable function f, so that the new function is not differentiable at c.

Take $f(x) = \begin{cases} 1, & \text{for } x = c \\ 0, & \text{for } x \neq c \end{cases}$. Then $f'_s(c) = \lim_{h\to 0}\dfrac{f(c+h)-f(c-h)}{2h} = 0$.
But $f'(c)$ doesn't exist since f is not continuous at $x = c$.

2. If $f \equiv 0$, then $f'(0) = 0 \neq 1$, contradiction arises. Hence $\exists\, x_0 \in \mathbb{R}$ s.t. $f(x_0) \neq 0$.
Then $f(x_0) = f(x_0 + 0) = f(x_0)f(0) \quad\Rightarrow\quad f(0) = 1$.
Also, f is differentiable at 0, hence $\lim_{h\to 0}\dfrac{f(h)-1}{h} = \lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = f'(0) = 1$.
Fix x. For all $h \neq 0$, $\dfrac{f(x+h)-f(x)}{h} = \dfrac{f(x)f(h)-f(x)}{h} = f(x)\dfrac{f(h)-1}{h}$
$\Rightarrow\quad f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = f(x)\lim_{h\to 0}\dfrac{f(h)-1}{h} = f(x)$.
Hence, f is differentiable on $\mathbb{R}$.