Tutorial 11 for MATH3270a

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In this tutorial, we will use an example to illustrate how to solve a 1^{st} order nonlinear system.

Consider the following system

$$\frac{dx}{dt} = (2+x)(y-x),$$
$$\frac{dy}{dt} = (4-x)(y+x);$$

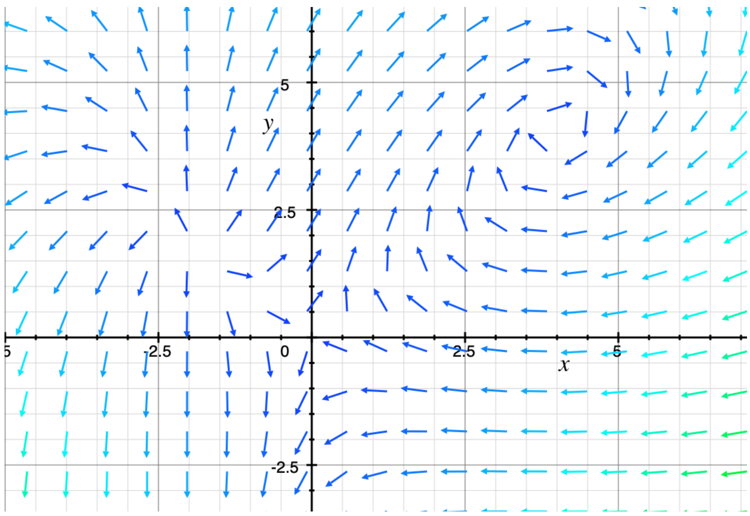
1. Find all the critical points of the system.

   Let $F = (2+x)(y-x), G = (4-x)(y+x)$, then by solving

   $$F = 0,$$
   $$G = 0,$$

   we find all critical points $P_1 = (0,0), P_2 = (-2,2), P_3 = (4,4)$.

2. Draw the direction field of the system.



3. Find the corresponding linear system near each critial points.

   Compute

   $$\frac{d(F,G)}{d(x,y)} = \begin{pmatrix} y-2x-2 & 2+x \\ 4-y-2x & 4-x \end{pmatrix}$$

   so the correponding linear system near each critical point is given by

   $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A(P_i)\begin{pmatrix} x \\ y \end{pmatrix}$$

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where

$$A(P_1) = \begin{pmatrix} -2 & 2 \\ 4 & 4 \end{pmatrix}, \quad A(P_2) = \begin{pmatrix} 4 & 0 \\ 6 & 6 \end{pmatrix}, \quad A(P_3) = \begin{pmatrix} -6 & 6 \\ -8 & 0 \end{pmatrix}.$$

4. Determine the type and stability of each critical points;

   The eigenvalues of $A(P_1)$ are $\lambda_1 = 1-\sqrt{17} < 0, \lambda_1 = 1+\sqrt{17} > 0$, so the critical point $P_1 = (0,0)$ is a saddle point and unstable. Moreover, the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 2 \\ 3-\sqrt{17} \end{pmatrix}, \quad r_2 = \begin{pmatrix} 2 \\ 3+\sqrt{17} \end{pmatrix}.$$

   The eigenvalues of $A(P_2)$ are $\lambda_1 = 4, \lambda_2 = 6$, so the critical point $P_2 = (-2,2)$ is a node and unstable. Moreover, the corresponding eigenvectors are

$$r_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

   The eigenvalues of $A(P_3)$ are $\lambda = -3 \pm \sqrt{39}i$, so the critical point $P_3 = (4,4)$ is a spiral point and stable. Note that the direction at point $(5,4)$ is $\begin{pmatrix} -6 \\ -40 \end{pmatrix}$, so the trajectory around $P_3 = (4,4)$ moves in clockwise direction to P_3.

5. Draw the phase portrait.