MATH 3270A Tutorial 8

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1 Sturm-Picone Comparison Theorem and its applications

Theorem 1 (Sturm-Picone Comparison Theorem). *Let $0 < \alpha_1(x) \leq \alpha_2(x)$ and $\beta_2(x) \leq \beta_1(x)$ be continuous functions on (a,b). Let y_1 and y_2 be solutions to the ODEs*

$$(\alpha_1(x)y_1'(x))' + \beta_1(x)y_1(x) = 0$$
$$(\alpha_2(x)y_2'(x))' + \beta_2(x)y_2(x) = 0$$

such that they are linearly independent. Then, between any consecutive zeros x_1, x_2 of y_2, there exists at least one zero of y_1.

Proof. Taking the difference between the product of the first equation and y_2 and the product of the second equation and y_1 gives

$$(\alpha_2 y_1 y_2' - \alpha_1 y_1' y_2)' = \alpha_2 y_1' y_2 - \alpha_1 y_1 y_2' + (\beta_1 - \beta_2) y_1 y_2 \tag{1}$$

Let $x_1 < x_2$ be consecutive zeros of y_2 and suppose there were no zeros of y_1 on (x_1, x_2). Then we have, by (1),

$$\left(\frac{y_2}{y_1}(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1')\right)' = (\beta_1 - \beta_2)y_2^2 + (\alpha_2 - \alpha_1)y_2'^2 + \alpha_2\left(y_2' - y_1'\frac{y_2}{y_1}\right)^2 \geq 0 \tag{2}$$

on (x_1, x_2). Integrating the equality gives

$$0 = \frac{y_2}{y_1}(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1')\bigg|_{x_1}^{x_2} = \lim_{x \to x_2}\left(\frac{y_2}{y_1}(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1')\right) - \lim_{x \to x_1}\left(\frac{y_2}{y_1}(\alpha_2 y_2' y_1 - \alpha_1 y_2 y_1')\right) \geq 0$$

This implies the RHS of (2) is identically zero, contradicting to the linear independence of y_1 and y_2. □

Here we present two applications of the theorem. The first one extends Theorem 2 in Tutorial 7.

Example 1. *Let $p(x) \geq 0$, $q(x) \leq 0$ be continuous functions on (a,b). Show that a non-trivial solution the following ODE has at most one zero.*

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

Solution

Let y be a non-trivial solution. Suppose on the contrary that y has two zeros $x_1 < x_2$. We may rewrite the equation as

$$(\alpha(x)y'(x))' + \beta(x)y = 0 \tag{3}$$

where $\alpha(x) = e^{\int_0^x p(s)ds} \geq 1$ and $\beta(x) = \alpha(x)q(x) \leq 0$ Consider the equation

$$y''(x) = 0 \tag{4}$$

By Sturm-Picone Comparison Theorem, for each solution y_2 of (4), there exists a $x_0 \in (x_1, x_2)$ of (4) such that $y_2(x_0) = 0$. This is a contradiction since a non-zero constant could be a solution to (4).

Example 2. *Let y be a solution of*

$$y'' + (2 + \sin x)y = 0$$

Show that there exists infinitely many zeros of y and the difference between any two consecutive zeros of y is less than π.

Solution

Consider the following equation

$$w'' + w = 0 \tag{5}$$

Note that $w = \sin(x - C)$ is a solution of the (5). By Sturm-Picone Comparison Theorem, any solutions of the original equation has at least one zero on $(k\pi + C, (k+1)\pi + C)$ for all $k \in \mathbb{N}$ and $C \in \mathbb{R}$.

2 Jacobi's formula

Theorem 2. *Let $A(t)$ be a matrix-valued function depending smoothly on $t \in (a, b)$. Then,*

$$\frac{d}{dt} \det A(t) = \det A(t) \mathbf{tr} \left(A^{-1} \frac{dA}{dt} \right)$$

Proof. Note that by the formula of Laplace 's expansion, we have

$$\frac{\partial \det(A)}{\partial a_{ij}} = (\mathbf{cof}(A))_{ij}$$

Hence, by Chain Rule, we have

$$\begin{aligned}
\frac{d}{dt} \det A(t) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \det(A)}{\partial a_{ij}} \frac{da_{ij}}{dt} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathbf{cof}(A))_{ij} \frac{da_{ij}}{dt} \\
&= (\mathbf{cof}(A)) : \frac{dA}{dt} \\
&= \mathbf{tr} \left((\mathbf{cof}(A))^T \frac{dA}{dt} \right) \\
&= \det A(t) \mathbf{tr} \left(A^{-1} \frac{dA}{dt} \right)
\end{aligned}$$

□