MATH 3270A Tutorial 6

Alan Yeung Chin Ching

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1 Higher-order linear ODEs with constant coefficients

Example 1. *Solve the following ODE.*

$$3y^{(5)} - 8y^{(4)} - 14y''' + 108y'' + 104y - 112y = 0$$

Solution

Consider the characteristic polynomial

$$\begin{aligned} 3r^5 - 8r^4 - 14r^3 + 108r^2 + 104r - 112 &= 0 \\ (r+2)^2(2r-3)(r^2 - 6r + 14) &= 0 \\ r = -2 \quad \text{or} \quad r = \frac{3}{2} \quad \text{or} \quad r = 3 - \sqrt{5}i \quad \text{or} \quad r = 3 + \sqrt{5}i \end{aligned}$$

Hence, the general solution of the ODE is given by

$$y = (A + Bx)e^{-2x} + Ce^{\frac{3}{2}x} + e^{3x}(D \sin \sqrt{5}x + E \cos \sqrt{5}x)$$

Example 2. *Determine the form of one particular solution of the following ODE.*

$$y^{(5)} - 6y^{(4)} + 16y''' - 32y'' + 48y' - 32y = e^x + x(x+1)e^{2x} + \sin 2x$$

Solution

Consider the characteristic polynomial

$$\begin{aligned} r^5 - 6r^4 + 16r^3 - 32r^2 + 48r - 32 &= 0 \\ (r-2)^3(r^2+4) &= 0 \\ r = 2 \quad \text{or} \quad r &= \pm 2i \end{aligned}$$

All homogeneous solutions are given by

$$y = (A + Bx + Cx^2)e^{2x} + D \sin 2x + E \cos 2x$$

Since xe^{2x}, x^2e^{2x} and $\sin 2x$ are homogeneous solutions, factors of the form x^m shall be added appropriately. Namely, the form of one particular solution should be given by

$$y = Ae^x + (B_1x^3 + B_2x^4 + B_3x^5)e^{2x} + x(C_1 \sin x + C_2 \cos x)$$

Example 3. *Let n be a positive integer. Solve the ODE*

$$\frac{d^ny}{dx^n} - y = 0$$

Solution

Consider the characteristic polynomial

$$r^n - 1 = 0$$

All the roots of the characteristic polynomial are given by roots of unity, namely

$$r = e^{\frac{2\pi ik}{n}}, \quad k = 0, \cdots, n-1$$

Hence, all solutions are given by

$$y = \begin{cases} A_0 e^x + B_0 e^{-x} + \\ \sum_{k=1}^{\frac{n-2}{2}} \exp\left(\sin\left(\frac{2k\pi i}{n}\right)x\right)\left(A_k \sin\left(\cos\left(\frac{2k\pi i}{n}\right)x\right) + B_k \cos\left(\cos\left(\frac{2k\pi i}{n}\right)x\right)\right) & \text{if } n \text{ is even} \\ A_0 e^x + \\ \sum_{k=1}^{\frac{n-1}{2}} \exp\left(\sin\left(\frac{2k\pi i}{n}\right)x\right)\left(A_k \sin\left(\cos\left(\frac{2k\pi i}{n}\right)x\right) + B_k \cos\left(\cos\left(\frac{2k\pi i}{n}\right)x\right)\right) & \text{if } n \text{ is odd} \end{cases}$$

2 The method of variation of parameters

Theorem 1 (The method of variation of parameters)**.** *Consider the ODE*

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-2}(x)y' + p_{n-1}(x)y = f(x) \tag{1}$$

where $p_i(x)$ and $f(x)$ are continuous functions on (a,b) for $1 \le i \le n-1$. Let $\{y_1, y_2, \cdots, y_n\}$ be a set of fundamental solutions of the corresponding homogeneous equation of (1). Then, a particular solution is given by

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n y_n(x)$$

with

$$u_i(x) = \int_0^x \frac{W_i(s)}{W(y_1, y_2, \cdots, y_n)(s)} ds$$

for $1 \le i \le n$ and

$$W_i(s) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{i-1} & 0 & y_{i+1} \cdots y_n \\ y_1' & y_2' & \cdots & y_{i-1}' & 0 & y_{i+1}' \cdots y_n' \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots \ddots \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{i-1}^{(n-1)} & f(s) & y_{i+1}^{(n-1)} \cdots y_n^{(n-1)} \end{vmatrix}$$

Remark.

$$\begin{aligned} y_p(x) &= u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n y_n(x) \\ &= \int_0^x \frac{W_1(s)y_1(x) + W_2(s)y_2(x) + \cdots + W_n(s)y_n(x)}{W_(y_1, \cdots, y_n)(s)} ds \\ &= \int_0^x \frac{\bar{W}(s;x)f(s)}{W_(y_1, \cdots, y_n)(s)} ds \end{aligned}$$

with

$$\bar{W}(s;x) = \begin{vmatrix} y_1(s) & y_2(s) & \cdots & y_n(s) \\ y_1'(s) & y_2'(s) & \cdots & y_n'(s) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(x) & y_2(x) & \cdots & y_n(x) \end{vmatrix}$$

The function

$$G(s;x) = \frac{\bar{W}(s;x)}{W_(y_1, \cdots, y_n)(s)}$$

is sometimes called the Green's function associated with the equation. It can be characterised by the following properties.

$$\left.\frac{\partial^k G(s,x)}{\partial x^k}\right|_{x=s} = 0 \quad \textbf{for} \quad k = 0, 1, \cdots, n-2$$

$$\left.\frac{\partial^{n-1} G(s,x)}{\partial x^{n-1}}\right|_{x=s} = 1$$

$$G(s,\cdot)^{(n)} + p_1(x)G(s,\cdot)^{(n-1)} + \cdots + p_{n-2}(x)G(s,\cdot)' + p_{n-1}(x)G(s,\cdot) = 0$$

for all $s \in (a,b)$.