MATH 3270A Tutorial 4

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1 The Method of Undetermined Coefficients

Consider the following ODE with constant-coefficients.

$$ay'' + by' + cy = r(x)$$

We may use the following table to find a particular solution, provided that y_p is not a homogeneous solution. The method is due to the observation that the derivatives of $r(x)$ of any orders have a general form.

$r(x)$	y_p
$Ae^{\alpha x}$	$ae^{\alpha x}$
$A_0 + A_1 x + \cdots + A_l x^l$	$a_0 + a_1 x + \cdots + a_l x^l$
$A\cos\alpha x + B\sin\alpha x$	$a\cos\alpha x + b\sin\alpha x$
$e^{\alpha x}(A_0 + A_1 t + \cdots + A_l t^l)$	$e^{\alpha x}(a_0 + a_1 t + \cdots + a_l t^l)$
$e^{\alpha x}(A\cos\alpha x + B\sin\alpha x)$	$e^{\alpha x}(a\cos\alpha x + b\sin\alpha x)$

Example 1. *Find the general solutions of the following ODE.*

$$2y'' + 11y' - 21y = 55e^{4x}$$

Solution

We first solve the corresponding homogeneous equations and we have the homogeneous solutions are given by

$$C_1 e^{\frac{3}{2}x} + C_2 e^{-7x}$$

Note that e^{4x} *is not a homogeneous solutions, we may suppose* $y = A\mathrm{e}^{4x}$ *is a particular solution of the equations and we solve* $A = 1$*. Hence, the general solutions are given by*

$$y = e^{4x} + C_1 e^{\frac{3}{2}x} + C_2 e^{-7x}$$

Example 2. *Find the general solutions of the following ODE.*

$$y'' + 4y' - 21y = e^{3x}$$

Solution

We first solve the corresponding homogeneous equations and we have the homogeneous solutions are given by

$$C_1 e^{-7x} + C_2 e^{3x}$$

Note that e^{3x} *is a homogeneous solution. Hence, we should look for solution that has the from* Axe^{3x}*. We then calculate that* $A = \frac{1}{10}$*. Hence, the general solutions are given by*

$$y(x) = C_1 e^{-7x} + C_2 e^{3x} + \frac{1}{10} x e^{3x}$$

Example 3. *Find the general solutions of the following ODE.*

$$y'' + y = \sin x + \cos x$$

Solution

We first solve the corresponding homogeneous equations and we have the homogeneous solutions are given by

$$C_1 \sin x + C_2 \cos x$$

Note that $\sin x + \cos x$ *is a homogeneous solution. Hence, we should look for solution that has the from* $Ax \sin x + Bx \cos x$*. We then calculate that* $A = \frac{1}{2}, B = -\frac{1}{2}$*. Hence, the general solutions are given by*

$$y(x) = C_1 \sin x + C_2 \cos x + \frac{1}{2} x \sin x - \frac{1}{2} x \cos x$$

Example 4. *Find the general solution of the following ODE.*

$$y'' + 6y' + 9y = 4e^{-3x}$$

Solution

We first solve the corresponding homogeneous equation and we have the homogeneous solutions are given by

$$C_1 e^{-3x} + C_2 x e^{-3x}$$

Note that e^{-3x} *is a homogeneous solution. Hence, we should look for solution that has the from* Axe^{-3x}*. We then calculate that* $A = 2$*. Hence, the general solutions are given by*

$$y(x) = C_1 e^{-3x} + C_2 x e^{-3x} + 2xe^{-3x}$$

Example 5. *Find the general solutions of the following ODE.*

$$y'' + 11y' - 12y = e^{-x} \sin 2x$$

Solution

We first solve the corresponding homogeneous equations and we have the homogeneous solutions are given by

$$C_1 e^{-12x} + C_2 e^{x}$$

Now we suppose that $y = e^x(A \sin 2x + B \cos 20)$ *is a particular solution of the equations and we solve* $A = -\frac{13}{500}$ *and* $B = \frac{9}{500}$*. Hence, the general solutions are given by*

$$y(x) = C_1 e^{-12x} + C_2 e^{x} - \frac{13}{500} e^{-x} \sin 2x - \frac{9}{500} e^{-x} \cos 2x$$

Example 6. *Find the general solutions of the following ODE.*

$$y'' + 5y' + 6y = e^{-x} + x^2$$

Solution

We first solve the corresponding homogeneous equations and we have the homogeneous solutions are given by

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x}$$

By superposition principle, we may solve the equation with inhomogeneous terms e^{-x} *and* x^2 *respectively and then add the solutions together.*
We suppose $y = Ae^{4x}$ *is a particular solution of the equations with inhomogeneous term* e^{-x} *and we solve* $A = \frac{1}{2}$*. Similarly, We suppose* $y = Bx^2 + Cx + D$ *is a particular solution of the equations with inhomogeneous term* x^2 *and we solve* $B = \frac{1}{6}, C = -\frac{5}{18}$ *and* $D = \frac{19}{108}$*.*

Hence, the general solutions are given by

$$y(x) = C_1 e^{-3x} + C_2 e^{-2x} + \frac{x^2}{6} - \frac{5x}{18} + \frac{e^{-x}}{2} + \frac{19}{108}$$

2 Euler' Equidimensional Equations

Definition 1. *n^{th} order (homogeneous) Euler' Equidimensional Equations are ODEs that have the form*

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0$$

where a_i are constants for $1 \leq i \leq n$ and $a_n \neq 0$.

Example 7. *Find the general solutions of the following ODE.*

$$2x^2 y'' - xy' + y = 0$$

Solution

We guess the one solution has the form x^m for some $m \in \mathbb{R}$. Then,

$$\begin{aligned} (2m(m-1) - m + 1)\, x^m &= 0 \\ 2m^2 - 3m + 1 &= 0 \\ m = 1 \text{ OR } m &= \frac{1}{2} \end{aligned}$$

Hence, the general solutions are given by $C_1 x + C_2 \sqrt{x}$ for $x > 0$

Exercise. *What if $x < 0$?*

Example 8. *Find the general solution of the following ODE.*

$$x^2 y'' - 3xy' + 4y = 0$$

Solution

Again, we guess the one solution has the form x^m for some $m \in \mathbb{R}$. Then,

$$\begin{aligned} (m(m-1) - 3m + 4)\, x^m &= 0 \\ m(m-1) - 3m + 4 &= 0 \\ m^2 - 4m + 4 &= 0 \\ m &= 2 \end{aligned}$$

To find another solution, we use the method of reduction of order. Note that by Abel's Theorem, we have the Wronskian $W(t)$ satisfies

$$W'(x) - \frac{3}{x} W(x) = 0$$

which has general solution $W(t) = Cx^3$. Hence, we have for any solution y_2 of the ODE,

$$\begin{aligned} y_1 y_2' - y_1' y_2 &= Cx^3 \\ \frac{d}{dx}\left(\frac{y_2}{y_1}\right) &= \frac{C}{x} \\ y_2 &= C y_1 \log x + C_2 y_1 \end{aligned}$$

Hence, the general solutions are given by $C_1 x^2 + C_2 x^2 \log x$ for $x > 0$

Exercise. *What if we have complex roots?*