MATH 3270A Tutorial 3

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1 Abels' Theorem and Reduction of order

Example 1. *It is given that one solution of the following ODE*

$$y'' - y' + e^{2t}y = 0$$

is $y_1 = \sin e^x$. *Find the general solutions to the ODE.*

Solution

Note that by Abel's Theorem, we have the Wronskian $W(t)$ *satisfies*

$$W'(t) - W(t) = 0$$

which has general solution $W(t) = Ce^t$. *Hence, we have for any solution* y_2 *of the ODE,*

$$\begin{aligned} y_1y_2' - y_1'y_2 &= Ce^t \\ \sin e^t y_2' - e^t \cos e^x y_2 &= Ce^t \\ y_2' - e^t \cot e^t y_2 &= Ce^t \csc e^t \\ \frac{d}{dx}(e^{-\log|\sin e^t|}y) &= Ce^{t-\log|\sin e^x|} \csc e^t \\ \frac{d}{dx}\left(\frac{y_2}{\sin e^t}\right) &= \frac{Ce^t}{\sin e^x} \csc e^t = Ce^t \csc^2 e^t \\ \frac{y_2}{\sin e^t} &= -C \cot e^t + C' \\ y_2 &= -C \cos e^t + C' \sin e^t \end{aligned}$$

Hence, all solutions of the ODE are given by $y = C_1 \cos e^t + C_2 \sin e^t$

Remark. *The equation can actually be solved directly by letting* $x = e^t$.

Example 2. *Show that* $y = e^x$ *and* $y = \sin x$ *cannot simultaneously be the solutions of* $y'' + p(x)y' + q(x)y = 0$, *where* $p(x)$ *and* $q(x)$ *are continuous function on* $(0, \pi)$.

Solution

We calculate the Wronskian $W(x)$ *associated with* $y_1 = e^x$ *and* $y_2 = \sin x$.

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^x & e^x \\ \sin x & \cos x \end{vmatrix} \\ &= e^x(\sin x - \cos x) \end{aligned}$$

which is zero for $x = \frac{\pi}{4}$ *and non-zero for* $x \neq \frac{\pi}{4}$ *on* $(0, \pi)$. *According to Abel's Theorem,* $y = e^x$ *and* $y = \sin x$ *cannot simultaneously be the solutions of the ODE.*

2 Second order linear ODEs with constant coefficients

Example 3. *Solve the following initial value problem.*

$$\begin{cases} y'' - 4y' + 53y &= 0 \\ y(0) &= 8 \\ y'(0) &= 2 \end{cases}$$

Solution

We solve the characteristic polynomial.

$$r^2 - 4r + 53 = 0$$
$$r = 2 + 7i \ \texttt{or}\ r = 2 - 7i$$

Hence, $e^{2t}\cos 7t$ *and* $e^{2t}\sin 7t$ *are two fundamental solutions and the general solutions are given by*

$$C_1 e^{2t}\cos 7t + C_2 e^{2t}\sin 7t$$

Putting the initial value, we have the $C_1 = 8$ *and* $C_2 = -2$

3 Factorization of Operators

Here we present an alternative method that does not require guessing the form of the solution.

Example 4. *Find the general solutions to the following ODE.*

$$\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 35y = 0$$

Solution

Note that we can factorize the LHS as

$$\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 35y = \left(\frac{d}{dx} - 5\right)\left(\frac{d}{dx} - 7\right)y$$

Here, multiplication of the operators represents composition of the operators. Hence,

$$\begin{aligned}
\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 35y &= 0 \\
\left(\frac{d}{dx} - 5\right)\underbrace{\left(\frac{d}{dx} - 7\right)y}_{=u} &= 0 \\
\frac{du}{dx} - 5u &= 0 \\
u &= C_1 e^{5u} \\
\frac{dy}{dx} - 7y &= C_1 e^{5x} \\
\frac{d}{dx}\left(e^{-7x}y\right) &= C_1 e^{-2x} \\
e^{-7x}y &= C_2 e^{-2x} + C_3 \\
y &= C_2 e^{5x} + C_3 e^{7x}
\end{aligned}$$

Example 5. *Find the general solutions to the following ODE.*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Solution

$$
\begin{aligned}
\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y &= 0 \\
\left(\frac{d}{dx} - 3\right)^2 y &= 0 \\
\frac{dy}{dx} - 3y &= C_1 e^{3x} \\
\frac{d}{dx}\left(e^{-3x} y\right) &= C_1 \\
e^{-3x} y &= C_1 t + C_2 \\
y &= (C_1 t + C_2) e^{3y}
\end{aligned}
$$

The method sometimes works also for variable-coefficient differential equation.

Example 6. *Find the general solutions to the following ODE.*

$$
\frac{d^2y}{dx^2} - (1 + \cot x)\frac{dy}{dx} + (\cot x)y = 0, \quad 0 < x < \pi
$$

Solution

$$
\begin{aligned}
\frac{d^2y}{dx^2} - (1 + \cot x)\frac{dy}{dx} + (\cot x)y &= 0 \\
\left(\frac{d}{dx} - \cot x\right)\left(\frac{d}{dx} - 1\right) y &= 0 \\
\frac{dy}{dx} - y &= C_1 \sin x \\
\frac{d}{dx}(e^{-x} y) &= C_1 e^{-x} \sin x \\
e^{-x} y &= C_2 e^{-x}(\sin x + \cos x) + C_3 \\
y &= C_2(\sin x + \cos x) + C_3 e^{x}
\end{aligned}
$$