MATH 3270A Tutorial 2

Alan Yeung Chin Ching

20th September 2018

1 Separable equations

Example 1 (Logistic Model)**.**

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N, \quad N > 0$$

where r and K are positive constants.

Solution[1]

If $N(t) = K$ for some t, then $N(t) = K$ for all t by uniqueness of the solution (See Theorem 1 below) . Hence, without loss of generality, we assume $N(t) \neq K$ for all t.

$$\begin{aligned}
\frac{dN}{dt} &= r\left(1 - \frac{N}{K}\right)N \\
\frac{dN}{\left(1 - \frac{N}{K}\right)N} &= r\,dt \\
\left(\frac{1}{N} + \frac{1}{K\left(1 - \frac{N}{K}\right)}\right)dN &= r\,dt \\
\log N - \log\left|1 - \frac{N}{K}\right| &= rt + C \\
\log\left|\frac{N}{1 - \frac{N}{K}}\right| &= rt + C \\
\frac{N}{1 - \frac{N}{K}} &= Ae^{rt} \qquad \textit{where } A = \pm e^{C} \\
N &= \frac{AKe^{rt}}{K + Ae^{rt}} = \frac{AK}{A + Ke^{-rt}} \\
&= \frac{N(0)K}{N(0) + (K - N(0))e^{-rt}}
\end{aligned}$$

Exercise. *The ODE is a Bernoulli Equation with $n = 2$. Try to solve the ODE by the method introduced in Tutorial 1.*

[1] I have implicitly assumed $0 < N < K$ for the calculation I presented in the tutorial. The solution here (with only slight modification) does not need this assumption.

2 Exact equations

Example 2.

$$\frac{\sin x}{y}\frac{dy}{dx} = -\frac{\sin x}{x} - \cos x \log xy, \quad 0 < x < \pi, y > 0$$

Solution

$$\begin{aligned} \frac{\sin x}{y}\frac{dy}{dx} &= -\frac{\sin x}{x} - \cos x \log xy \\ \frac{\sin x}{y}\frac{dy}{dx} + \frac{\sin x}{x} + \cos x \log xy = 0 &= M\frac{dy}{dx} + N \end{aligned}$$

Now, note that

$$\frac{\partial M}{\partial x} = \frac{\cos x}{y} = \frac{\partial N}{\partial y}$$

Hence, the equation is exact. Let f *be such that*

$$\frac{\partial f}{\partial y} = M \qquad \frac{\partial f}{\partial x} = N$$

Integrating M *with respect to* y *gives*

$$f(x, y) = \sin x \log y + g(x).$$

for some function g*. Differentiating with respect to* x *once gives*

$$\begin{aligned} N &= \cos x \log y + g'(x) \\ \implies g'(x) &= \frac{\sin x}{x} + \cos x \log x \\ \implies g(x) &= \sin x \log x + C \end{aligned}$$

Take $C = 0$ *and we have*

$$f(x, y) = \sin x \log y + \sin x \log x = \sin x \log(xy)$$

Hence, we have

$$\begin{aligned} \frac{d}{dx}\left(\sin x \log(xy)\right) &= 0 \\ \sin x \log(xy) &= C' \\ y &= \frac{1}{x}e^{\frac{C'}{\sin x}} \end{aligned}$$

3 Integrating factor for non-linear equations

Example 3.

$$(2x + 5y)\frac{dy}{dx} + y = 0$$

Solution

Let M and N be such that

$$(2x+5y)\frac{dy}{dx}+y=N\frac{dy}{dx}+M$$

We compute that

$$\begin{aligned}\frac{\partial N}{\partial x}&=2\\ \frac{\partial M}{\partial y}&=1\\ \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}&=\frac{1}{y}\end{aligned}$$

Hence, one integrating factor is given by finding one solution to $\mu'(y)=\frac{\mu(y)}{y}$, one may choose

$$\mu(y)=y$$

The ODE becomes

$$\begin{aligned}\left(2xy+5y^2\right)\frac{dy}{dx}+y^2&=0\\ \frac{d}{dx}\left(xy^2+\frac{5}{3}y^3\right)&=0\\ xy^2+\frac{5}{3}y^3&=C\end{aligned}$$

4 General first-order ODEs: Existence and Uniqueness

Theorem 1 (The Fundamental Theorem of ODEs, Picard - Lindelf)**.** *Let $f:(a,b)\times(c,d)\to\mathbb{R}$ be a continuous function. Suppose $\frac{\partial f}{\partial y}$ is continuous on $(a,b)\times(c,d)$. Then, for all $t_0\in(a,b)$ and $y_0\in(c,d)$, there exists $\delta>0$ such that the following initial value problem*

$$\begin{cases}\frac{dy}{dt}&=f(t,y(t))\\ y(t_0)&=y_0\end{cases}\tag{1}$$

has a unique solution in $(t_0-\delta,t_0+\delta)\subseteq(a,b)$.

Remark.

1. *For fixed (t_0,y_0), the condition $\frac{\partial f}{\partial y}$ is continuous on $(a,b)\times(c,d)$ can actually be weakened to $f(t,\cdot)$ is uniformly Lipschitz continuous in t, i.e. there exists $L>0$ such that $|f(t,y_1)-f(t,y_2)|\le L|y_1-y_2|$ for all t close to t_0 and y_1,y_2 close to y_0.*
2. *For linear equations, the existence and uniqueness can be obtained in the whole domain (a,b) by Theorem 1 in tutorial 1.*
3. *If the condition on the continuity of $\frac{\partial f}{\partial y}$ on $(a,b)\times(c,d)$ is not satisfied, the existence of solution is still true.*
4. *There is no similar result in the theory of partial differential equations.*