MATH 3270A Tutorial 10

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1 Variation of Parameters

Theorem 1. *Let $A(t), r(t)$ be matrix-valued and vector-valued functions depends continuously on $t \in (a, b)$. Consider the system of first-order ODE*

$$y'(t) = A(t)y(t) + r(t) \tag{1}$$

Let $X(t)$ be a fundamental matrix of the corresponding homogeneous system of (1). Then, a particular solution of (1) is given by

$$y(t) = X(t) \int X^{-1}(t)r(t)dt$$

Example 1. *Solve the following system of ODEs.*

$$y'(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} y + \begin{pmatrix} e^t \\ 0 \end{pmatrix} \tag{2}$$

Solution

The homogeneous solutions are given by

$$y(t) = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Hence, a fundamental matrix is given by

$$X(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^t \end{pmatrix}$$

Hence, a particular solution is given by

$$
\begin{aligned}
y(t) &= X(t)\int X^{-1}\begin{pmatrix} e^t \\ 0 \end{pmatrix} dt \\
&= \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\int \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}^{-1}\begin{pmatrix} e^t \\ 0 \end{pmatrix} dt \\
&= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\int \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}\begin{pmatrix} e^t \\ 0 \end{pmatrix} dt \\
&= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\int \begin{pmatrix} 3 \\ -e^{2t} \end{pmatrix} dt \\
&= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} 3t \\ -e^{2t}/2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 3te^t - e^t/2 \\ 3te^t - 3e^t/2 \end{pmatrix}
\end{aligned}
$$

Hence, the general solutions are given by

$$
y = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3te^t - e^t/2 \\ 3te^t - 3e^t/2 \end{pmatrix}
$$

2 Phase Portrait

Example 2. *Draw a representative set of trajectory of the phase portrait of the following system.*

$$
y' = \begin{pmatrix} -5 & 8 \\ 2 & 1 \end{pmatrix} y
$$

Solution

Now, we find the eigenvalues and the corresponding eigenvectors of

$$
A = \begin{pmatrix} -5 & 8 \\ 2 & 1 \end{pmatrix}
$$

The characteristic polynomial is given by

$$
\det(A - \lambda I) = \det\begin{pmatrix} -5-\lambda & 8 \\ 2 & 1-\lambda \end{pmatrix} = -\lambda^2 - 4\lambda + 21
$$

which has roots $\lambda_1 = 3$ *and* $\lambda_2 = -7$. *Note that*

$$
v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}
$$

are corresponding eigenvectors.

Example 3. *Draw a representative set of trajectory of the phase portrait of the following system.*

$$y' = \begin{pmatrix} 6 & -2 \\ 5 & 4 \end{pmatrix} y$$

Solution

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 6 & -2 \\ 5 & 4 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(A - \lambda I) = \det \begin{pmatrix} 6-\lambda & -2 \\ 5 & 4-\lambda \end{pmatrix} = -\lambda^2 + 10\lambda - 33$$

which has roots $\lambda_1 = 5 + 3i$ *and* $\lambda_2 = 5 - 3i$*. Note that*

$$v_1 = \begin{pmatrix} 1+3i \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} 1-3i \\ 5 \end{pmatrix}$$

are corresponding eigenvectors.

Example 4. *Draw a representative set of trajectory of the phase portrait of the following system.*

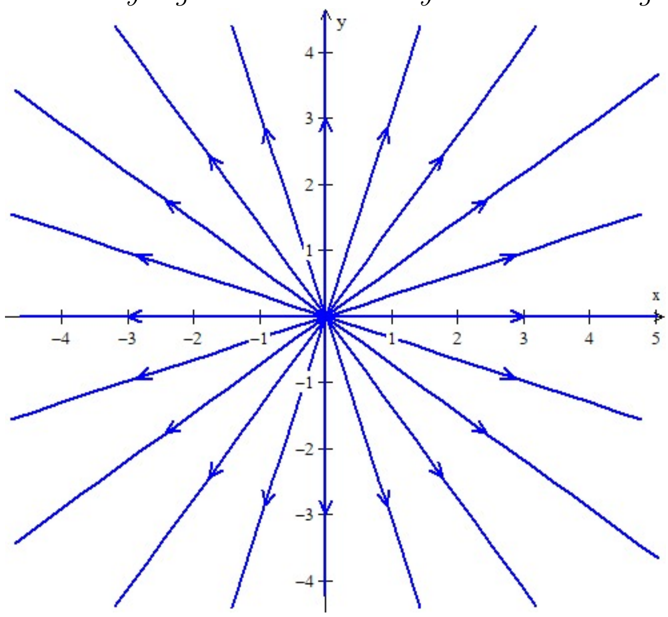
$$y' = \begin{pmatrix} -5 & 8 \\ 2 & 1 \end{pmatrix} y$$

Solution

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

Clearly, $\lambda = 9$ *is the only eigenvalue and every vector is an eigenvector.*



Example 5. *Draw a representative set of trajectory of the phase portrait of the following system.*

$$y' = \begin{pmatrix} 11 & -5 \\ 5 & 1 \end{pmatrix} y$$

Solution

Now, we find the eigenvalues and the corresponding eigenvectors of

$$A = \begin{pmatrix} 11 & -5 \\ 5 & 1 \end{pmatrix}$$

The characteristic polynomial is given by

$$\det(A - \lambda I) = \det \begin{pmatrix} 11 - \lambda & -5 \\ 5 & 1 - \lambda \end{pmatrix} = -\lambda^2 + 12\lambda - 36$$

which has double roots $\lambda = 6$. Note that

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

is a corresponding eigenvector. Now, we want to find two generalized eigenvectors. To do so, we solve for v_2 such that

$$(A - 6I)v_2 = v_1$$
$$\begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

One may take

$$v_2 = \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}$$

Example 6. *Draw a representative set of trajectory of the phase portrait of the following system.*

$$\begin{cases} y_1' &= 2y_1 + y_2 \\ y_2' &= 4y_1 + 2y_2 \end{cases}$$

Solution

The system is solved in the last tutorial and the solutions are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$