Suggested solutions to Homework 1 for MATH3270a

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1. (**4points=0.5points** $\times$ **8**) Solve the following initial-value problems:

   (a) $t^4y' + 5t^3y = e^{-t}, y(-1) = 0$ for $t < 0$;
   (b) $y' = \frac{y^2}{t}, y(1) = 3$;
   (c) $y + (2t - 3ye^y)y' = 0, y(1) = 0$;
   (d) $y' = ty^3(1+t^2)^{-1/2}, y(0) = 1$;
   (e) $y' = \frac{y-4t}{t-y}, y(1) = 3$ for $t > 0$;
   (f) $y' = y - 2y^2, y(1) = 0$;
   (f)' $y' = y - 2y^2, y(1) = 1$;
   (g) $(3t^2y + 2ty + y^3) + (t^2 + y^2)y' = 0, y(0) = 1$;
   (h) $(t^2 + 3ty + y^2) - t^2y' = 0, y(1) = 0$ for $t > 0$.

   Solution:

   (a) Multiplying the ODE by t gives

   $$\frac{d}{dt}(t^5y) = te^{-t},$$

   then integrating both sides we have

   $$t^5y = \int_{-1}^{t} se^{-s}ds$$

   where we have used the initial condition $y(-1) = 0$. Thus the solution is given by for $t < 0$

   $$y = -\frac{t+1}{t^5}e^{-t}.$$

   Remark: *It's noted that the maximal existence interval of the solution in this case is* $(0, \infty)$ *or* $(-\infty, 0)$*, which together with initial condition* $y(-1) = 0$ *implies that the solution is unique only for* $t < 0$.

   (b) This is a separable ODE. If $y \neq 0$, then rewrite the original ODE as

   $$\frac{dy}{y^2} = \frac{dt}{t}.$$

   It follows from direct integration that

   $$-\frac{1}{y} = \ln|t| + C$$

   with constant C. Together with initial data $y(1) = 3$, we can obtain the following solution

   $$y = -\frac{3}{3\ln t - 1}, \quad 0 < t < e^{\frac{1}{3}}.$$

   Remark: *By Existence and Uniqueness Theorem, this problem is uniquely solvable on some neighborhood of* $(1, 3)$.

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(c) The unique solution is $y \equiv 0$. It's trivial that 0 is a solution. In fact, we can show that the solution is unique by the following two ways:

Method 1: If $y \neq 0$, rewrite ODE as

$$\frac{dt}{dy} = -\frac{2}{y}t + 3e^y$$

which is a 1st order linear ODE of t as a function of y, so the solution is given by

$$e^{\int \frac{2}{y}} t = C + \int 3e^y e^{\int \frac{2}{y}}$$

that is,

$$ty^2 = C + 3(y^2 - 2y + 2)e^y \tag{1}$$

which together with initial data $y(1) = 0$ gives that

$$ty^2 = -6 + 3(y^2 - 2y + 2)e^y.$$

However, by Implicit Function Theorem, we cannot get a function $y(t)$ satisfying (1) and across the point $y(1) = 0$.

Method 2: Rewrite the ODE as

$$y' = -\frac{y}{2t - 3ye^y} =: f(t, y),$$

then $f(t, y)$ and $f_y(t, y)$ are continuous around some neighborhood of $(1, 0)$, so it's follows from Existence and Uniqueness Theorem that $y = 0$ is the unique solution.

(d) This is a separable ODE, so rewrite as

$$\frac{dy}{y^3} = \frac{tdt}{\sqrt{1 + t^2}},$$

then integrating both sides yields

$$-\frac{1}{2y^2} = \sqrt{1 + t^2} + C.$$

Then constant C is determined by $y(0) = 1$, so

$$y = \frac{1}{\sqrt{3 - 2\sqrt{1 + t^2}}}$$

for $-\frac{\sqrt{5}}{2} < t < \frac{\sqrt{5}}{2}$.

(e) It's noted that the source term $f(y, t) = \frac{y-4t}{t-y}$ is homogeneous with t, y, that is, $f(ky, kt) = f(y, t)$ for any constant k, so we consider the new variable

$$z = \frac{y}{t}.$$

Then $z' = \frac{y'}{t} - \frac{z}{t}$ or $y' = tz' + z$, so the original ODE becomes

$$tz' + z = \frac{z - 4}{1 - z}$$

or equivalently

$$\frac{1 - z}{z^2 - 4}dz = \frac{dt}{t}.$$

Integrating both sides gives

$$\frac{1}{4} \ln \left| \frac{1}{(z + 2)^3 (z - 2)} \right| = \ln |t| + C,$$

or equivalently

$$(y+2t)^3(y-2t) = C$$

Finally, by the initial condition $y(1) = 3$, we have

$$(y+2t)^3(y-2t) = 125. \tag{2}$$

Remark: *For this problem, we only give the implicit solution formula (3). It should be noted that the solution to Problem 1(d) satisfies (3), which can not imply that any $y(t)$ satisfying (3) is a solution to the original Problem 1(d).*

(f) It's noted that $y = 0$ is a solution and that $f(t, y) = y - 2y^2$ and $f_y = 1 - 4y$ are continuous on $\mathbb{R}^2$, then by Existence and Uniqueness Theorem, it's the unique one.[1]

(f)' This is a separable ODE, if $y - 2y^2 \neq 0$, rewrite as

$$\frac{dy}{y - 2y^2} = dt.$$

Then it follows from integrating both sides that

$$\frac{y}{1-2y} = Ce^t$$

with arbitrary constant C. Note that $y(1) = 1$ implies that $C = -e^{-1}$ and

$$y = \frac{e^{t-1}}{2e^{t-1} - 1}$$

for $t > 1 - \ln 2$.

(g) Let $M = 3t^2y + 2ty + y^3$ and $N = t^2 + y^2$, so $M_y = 3t^2 + 2t + 3y^2, N_t = 2t$. It's noted that

$$\frac{M_y - N_t}{N} = 3,$$

so it's promising to find an intergrating factor μ of the form $\mu = \mu(t)$. Then multiply the ODE by $\mu(t)$ such that it's exact, that is,

$$(\mu M)_y = (\mu N)_t,$$

so $\mu(t)$ satisfies

$$\mu' = \frac{M_y - N_t}{N}\mu = 3\mu,$$

which implies $\mu(t) = e^{3t}$. Since $(\mu M)_y = (\mu N)_t$, then there exists a function $\varphi(t, y)$ such that

$$\partial_t \varphi = \mu M = e^{3t}(3t^2y + 2ty + y^3), \tag{3}$$
$$\partial_y \varphi = \mu N = e^{3t}(t^2 + y^2). \tag{4}$$

By solving (4) firstly, we have

$$\varphi(t, y) = e^{3t}(t^2y + \frac{1}{3}y^3) + h(t) \tag{5}$$

with some function $h(t)$. Then insert (5) into (3), we have

$$h'(t) = 0$$

which implies that we can take $h = 0$. Finally, the general solution to the ODE is given by

$$\varphi = e^{3t}(t^2y + \frac{1}{3}y^3) = C.$$

The constant C is determined by $y(0) = 1$, so

$$e^{3t}(t^2y + \frac{1}{3}y^3) = \frac{1}{3}.$$

[1] You can also use **Method 1** in Problem 1(c) to show that $y = 0$ is the unique solution.

(h) It's noted that the ODE can be wrriten as

$$y' = \frac{t^2 + 3ty + y^2}{t^2}.$$

So if let $z = \frac{y}{t}$, then we have the following ODE

$$tz' + z = 1 + 3z + z^2$$

which is separable, so the solution is given by

$$-\frac{1}{z+1} = \ln|t| + C,$$

or equivalently

$$y + t = -\frac{t}{\ln|t| + C}.$$

Then by the initial condition $y(1) = 0$, we can obtain that

$$y = -t\frac{\ln t}{\ln t - 1}, \quad 0 < t < e.$$

2. (**2points=0.5points** $\times$ **4**) Determine whether each of the following equations is exact or not, if it is then find the solutions:

(a) $(e^t \sin y - 3y \sin t) + (e^t \cos y + 3\cos t)y' = 0$;

(b) $(t+2)\sin y + t\cos y y' = 0$;

(c) $\frac{t}{(t^2+y^2)^{3/2}} + \frac{y}{(t^2+y^2)^{3/2}}y' = 0$;

(d) $y' = \frac{ay+b}{cy+d}$.

Solution:

(a) It's exact. In fact, let $M = e^t \sin y - 3y\sin t$ and $N = e^t \cos y + 3\cos t$, it's easy to check that

$$M_y = e^t \cos y - 3\sin t = N_t.$$

Then by following the procedure in 1(g) [2], the general solution is given by

$$e^t \sin y + 3y\cos t = C,$$

where C is an arbitrary constant.

(b) It's not exact. In fact, let $M = (t+2)\sin y$ and $N = t\cos y$, then $M_y = (t+2)\cos y \neq \cos y = N_t$.

(c) It's exact. In fact, let $M = \frac{t}{(t^2+y^2)^{3/2}}$ and $N = \frac{y}{(t^2+y^2)^{3/2}}$, so

$$M_y = -\frac{3ty}{(t^2+y^2)^{5/2}} = N_t.$$

Then by following the procedure in 1(g), the general solution is given by

$$\frac{1}{(t^2+y^2)^{1/2}} = C$$

where C is a constant to be determined by additional condition.

[2] We omit here, please expand the details by yourself.

(d) It's exact iff $a = 0$. In fact, rewrite as

$$ay + b - (cy + d)y' = 0,$$

then let $M = ay + b$ and $N = -cy - d$, so

$$M_y = N_t \Leftrightarrow a = 0.$$

Let $a = 0$. It's noted that this is a separable ODE, then the general solution is given by

$$\frac{c}{2}y^2 + dy = bt + C$$

with arbitrary constant C.

3. (**2points**) Consider the general first order linear equation $y' = p(t)y + g(t)$, show that

- (**0.5points**) if $y_1(t)$ is a solution to $y' = p(t)y$, so is $cy_1(t)$ for any $c \in \mathbb{R}$;
- (**0.5points**) if $y_2(t)$ is a soluiton to $y' = p(t)y + g(t)$, so is $cy_1(t) + y_2(t)$ for any $c \in \mathbb{R}$;
- (**1point**) all the solutions to $y' = p(t)y + g(t)$ is of the form $cy_1(t) + y_2(t)$ for some $c \in \mathbb{R}$.

Solution:

- It's noted that for any $c \in \mathbb{R}$

$$\frac{d}{dt}(cy_1(t)) = cy_1'(t) = cp(t)y_1(t),$$

so $cy_1(t)$ is a solution to $y' = p(t)y$.

- It's noted that for any $c \in \mathbb{R}$

$$\frac{d}{dt}(cy_1(t) + y_2(t)) = cy_1'(t) + y_2'(t) = cp(t)y_1 + p(t)y_2 + g(t) = p(t)(cy_1 + y_2) + g(t)$$

so $cy_1 + y_2$ is a solution to $y' = p(t)y + g(t)$.

- Let $y_1 \neq 0$ be a solution to $y' = p(t)y$, we first **claim** that all solutions to $y' = p(t)y$ are of the form $y_c = cy_1$ for some constant $c \in \mathbb{R}$. In fact, consider $\frac{y_c}{y_1}$, it's easy to get that

$$\frac{d}{dt}(\frac{y_c}{y_1}) \equiv 0,$$

which implies that for any solution y_c to $y' = p(t)y$ there exists some constant c such that $y_c = cy_1$.
Next, for any solution y to $y' = p(t)y + g(t)$, it's noted that $z = y - y_2$ satisfies

$$z' = p(t)z.$$

Then by the above claim, we know that there exists some $c \in \mathbb{R}$ such that $y - y_2 = cy_1$.

4. (**2points**) Consider the differential equation

$$M(t, y) + N(t, y)y' = 0.$$

Assume that we have $tM - yN \neq 0$, and the fraction $\frac{N_t - M_y}{tM - yN} = R(ty)$ depending only on the quantity ty only, then **show (1point)** that the above differential equation has an integrating factor of the form $\mu(ty)$ and **find (1point)** a general formula for this integrating factor.

Solution: Multiplying the ODE by $\mu(ty)$ gives that

$$\mu(ty)M(t, y) + \mu(ty)N(t, y)y' = 0,$$

then let $z = ty$, it's noted that

$$(\mu M)_y = (\mu N)_t \Leftrightarrow \mu'(z) = \mu(z)\frac{N_t - M_y}{tM - yN} = \mu(z)R(z).$$

Now by solving $\mu'(z) = \mu(z)R(z)$, we get

$$\mu = Ce^{\int R(z)dz} = Ce^{\int^{ty} R(z)dz} \tag{6}$$

where $C \neq 0$ is an arbitrary constant. Thus in this way, we do find an intergrating factor with form $\mu(ty)$, more presicely, given by the fomula (6).