MATH 3270A Tutorial 1

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1 Basic concepts

Classify the following ODEs by their order, linearity.

1. $y'' = x^2 y + \sin x$ Ans: second-order linear (inhomogeneous) ODE
2. $y''' = y^2 + 2y + \frac{1}{y}$ Ans: third-order non-linear ODE

2 First-order linear ODEs: the method of integrating factor

Theorem 1. *Let p and q be continuous functions on $I = (a, b)$. Then all the solutions of the ODE*

$$y'(t) = p(t)y(t) + q(t) \tag{1}$$

are given by

$$y(t) = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)dt + C\right), C \in \mathbb{R}$$

where

$$\mu(t) = e^{-\int p(t)dt}$$

is called an integrating factor of (1).

Solve the general solution of the following ODEs.

1. $y' = 10y$ Ans: $y = Ce^{10x}$
2. $xy' = y + 12x$ Ans: $y = Cx + 12x \log x$
3. $y' = \frac{y + 7x^2 \sin x}{x}$ Ans: $y = Cx - 7x \cos x$

3 Bernoulli Equations

Definition 1. *Bernoulli Equations are non-linear ODEs that have the form*

$$\frac{dy}{dx} = p(x)y + q(x)y^n$$

for some $n \in \mathbb{R} \setminus \{0, 1\}$ and continuous functions p, q.

Bernoulli Equations could be transformed to first-order linear ODEs by $w = \frac{1}{y^{n-1}}$.

Example. *Solve the following initial value problem*

$$\begin{cases} y'(t) &= \dfrac{y(t)}{t} + [y(t)]^3 \\ y(1) &= -\frac{1}{2} \end{cases}$$

Solution

$$\begin{aligned}
y'(t) &= \frac{y(t)}{t} + [y(t)]^3 \\
\frac{y'(t)}{[y(t)]^3} &= \frac{y(t)}{t[y(t)]^3} + 1 \quad (**) \\
-\frac{1}{2}\frac{d}{dt}\left(\frac{1}{[y(t)]^2}\right) &= \frac{1}{t[y(t)]^2} + 1 \\
\frac{d}{dt}\left(\frac{1}{[y(t)]^2}\right) &= -\frac{2}{t}\left(\frac{1}{[y(t)]^2}\right) - 2 \\
\frac{d}{dt}\left(\frac{t^2}{[y(t)]^2}\right) &= -2t^2 \\
\frac{t^2}{[y(t)]^2} &= -\int 2t^2 dt + C = -\frac{2t^3}{3} + C \\
\frac{1}{[y(t)]^2} &= -\frac{2t}{3} + \frac{C}{t^2} \\
y(t) &= \pm\sqrt{\left(\frac{C}{t^2} - \frac{2t}{3}\right)^{-1}} = \pm\frac{\sqrt{3}t}{\sqrt{3C - 2t^3}}
\end{aligned}$$

Putting the initial value, we have

$$y(t) = -\frac{\sqrt{3}t}{\sqrt{14 - 2t^3}}$$

4 Application to differential inequality

Here comes another interesting and important application of the method of integrating factor, which is based on the simple observation that an integrating factor for a first order linear ODE has a definite sign.

Example (Gronwall inequality)**.** *Let $y : [0,1] \to \mathbb{R}$ be a non-negative differentiable function with $y(0) = 0$. Suppose there exists a continuous function ϕ on $[0,1]$ and such that*

$$\frac{dy}{dx} \leq \phi(x)y(x)$$

for all $x \in [0,1]$. Show that $y \equiv 0$.

Solution:

We try to "solve" the differential inequality by the method of integrating factor. Let

$$\mu(x) = e^{-\int \phi(x)dx} > 0$$

[0](**)I should have justified this step more carefully in the tutorial. Note that the initial value is non-zero, hence the solution is non-zero in a small neighbourhood of 1 (by the continuity of the solution). After all, we are solving the equation near $t = 1$, and consequently we are allowed to divide the equation by $[y(t)]^3$. What's more, if the solution attains zero for some $t > 0$, we have by uniqueness argument, the solution is zero in a neighbourhood of that point. This, together with a continuity argument, implies the solution in constant zero for all $t > 0$. Note, however, a solution can be zero at $t = 0$ without being constant zero for $t > 0$. Let me also thank the student who raised this question after the tutorial.

denote an integrating factor. Then, we have

$$
\begin{aligned}
\frac{dy}{dx} &\leq \phi(x)y(x) \\
\frac{dy}{dx} - \phi(x)y(x) &\leq 0 \\
\mu(x)\left(\frac{dy}{dx} - \phi(x)y(x)\right) &\leq 0 \\
\frac{d}{dx}\left(\mu(x)y(x)\right) &\leq 0 \\
\int_0^\theta \frac{d}{dx}\left(\mu(x)y(x)\right)dx &\leq \int_0^\theta 0dx = 0 \qquad \forall\theta \in (0,1] \\
\mu(x)y(x)|_0^\theta = \mu(\theta)y(\theta) &\leq 0 \\
y(\theta) &\leq 0
\end{aligned}
$$

The last inequality forces $y(\theta) = 0$ *for all* $\theta \in [0,1]$.

More generally, we have

Exercise. *Let* $y : [0,1] \to \mathbb{R}$ *be non-negative differentiable functions with* $y(0) = 0$. *Suppose there exist non-negative continuous functions* ϕ *and* ξ *on* $[0,1]$ *such that*

$$\frac{dy}{dx} \leq \phi(x)y(x) + \xi(x)$$

for all $x \in [0,1]$. *Then, for all* $x \in [0,1]$

$$y(x) \leq e^{\int_0^x \phi(s)ds}\left(y(0) + \int_0^x \xi(s)ds\right)$$