HOMEWORK III (DEADLINE : 9TH NOVEMBER, 2018)

ORDINARY DIFFERENTIAL EQUATIONS

Answer all questions:

(1) (4 points)**Find** the general solution to the following differential equations:

(a) $y^{(4)} + 2y^{(3)} + y^{(2)} = 0$;

(b) $y^{(4)} + 8y^{(2)} + 16y = 0$;

(c) $y^{(3)} + y^{(2)} + y^{(1)} + y = 2e^{-t} + 4t$;

(d) $y^{(4)} + 2y^{(2)} + y = 4 + \cos(2t)$.

(2) (1 point) **Determine** a suitable form of $Y(t)$ for using the method of undetermined coefficients to the following equations (you don't have to solve the equation explicitly, but simply guess the form of $Y(t)$ to be used):

(a) $y^{(3)} - 2y^{(2)} + y^{(1)} = 3t^3 + 2e^t$;

(b) $y^{(4)} - y^{(3)} - y^{(2)} + y^{(1)} = t^2 + 8 + t\sin(t)$.

(3) (2 point) **Write** down a formula involving integrals for a particular solution $Y(t)$ of the differential equation

$$y^{(3)} - 3y^{(2)} + 3y^{(1)} - y = r(t),$$

and use it to **solve** for $Y(t)$ when $r(t) = t^{-2}e^t$.

(4) (3 points) If y_1 is a solution to the equation

$$y^{(3)} + p_2(t)y^{(2)} + p_1(t)y^{(1)} + p_0(t)y = 0,$$

(a) use the substitution $y = y_1(t)v(t)$ to **derve** a second order ODE for $u = v'$;
(b) use it to **solve** for the general solution for

$$(2-t)y^{(3)} + (2t-3)y^{(2)} - ty^{(1)} + y = 0$$

for $t < 2$ provided that $y_1(t) = e^t$ is a solution.

(5) (5 points) [Method of Annihilators] In this question, we will consider another way of arriving at a guess for the form of a particular solution $Y(t)$ for solving a inhomogeneous linear equations. This start with the observation that any terms of the from $P_k(t) = A_0 + \cdots + A_k t^k$, or $e^{\alpha t}P_k(t)$ or $e^{\alpha t}\sin(\mu t)P_k(t)$ or $e^{\alpha t}\cos(\mu t)P_k(t)$ can be viewed as solution of certain linear homogeneous differential equations with constant coefficient.

For example, we can treat e^{-t} as a solution for the equation $(\frac{d}{dt}+1)y=0$, and in that case we say the differential operator $(\frac{d}{dt}+1)$ annihilate, or to be an annihilator of e^{-t}. In the same way, $\frac{d^2}{dt^2}+4$ is an annihilator of $\sin(2t)$ or $\cos(2t)$, and $(\frac{d}{dt}-3)^2=\frac{d^2}{dt^2}-6\frac{d}{dt}+9$ is an annihilator of e^{3t} or te^{3t}, and so forth.

To domenstrate how it works, we consider the following differential equation

$$(\frac{d}{dt}-2)^3(\frac{d}{dt}+1)y=3e^{2t}-te^{-t}, \tag{0.1}$$

and try to solve for a particular solution $Y(t)$.

(a) **Show** that the linear differential operators with *constant coefficients* obey the commutative law

$$(\frac{d}{dt}-a)(\frac{d}{dt}-b)f=(\frac{d}{dt}-b)(\frac{d}{dt}-a)f$$

for any twice-differentiable f and any constants a,b.

(b) **Show** that the operator $(\frac{d}{dt}-2)(\frac{d}{dt}+1)^2$ annihilates the terms on the R.H.S. of equation (0.1).

(c) By applying $(\frac{d}{dt}-2)(\frac{d}{dt}+1)^2$ to both side of equation (0.1), **show** that a particular solution $Y(t)$ should satisfy

$$(\frac{d}{dt}-2)^4(\frac{d}{dt}+1)^3Y=0,$$

and hence **show** that

$$Y(t)=c_1e^{2t}+c_2te^{2t}+c_3t^2e^{2t}+c_4t^3e^{2t}+c_5e^{-t}+c_6te^{-t}+c_7t^2e^{-t},$$

for some constant $c_1,c_2,c_3,c_4,c_5,c_6,c_7$ to be determined.

(d) Observe that e^{2t},te^{2t},t^2e^{2t} and e^{-t} are solutions to the homogeneous equation corresponding to equation (0.1), and hence these terms are not useful in solving for a particular solution. Therefore, we choose $c_1=c_2=c_3=c_5=0$ and **solve** a particular solution $Y(t)$ for equation (0.1).

(e) For each of $P_k(t)$, $e^{\alpha t}P_k(t)$, $e^{\alpha t}\sin(\mu t)P_k(t)$, $e^{\alpha t}\cos(\mu t)P_k(t)$, **write down** a polynomial $f(r)$ such that $f(\frac{d}{dt})$ is an annihilator for it (It is not necessary to expand the polynomial).

End of Homework 3