REVIEW OF HOLOMORPHIC FUNCTIONS

PO-LAM YUNG

Theorem 1. *Let Ω be an open set in $\mathbb{C}$, and $f\colon \Omega \to \mathbb{C}$ be a complex-valued function on Ω. Then the following are equivalent:*

(a) f is holomorphic on Ω;

(b) $f'(z) := \lim_{h\to 0} \dfrac{f(z+h)-f(z)}{h}$ exists for every $z \in \Omega$;

(c) $f = u + iv$ where u and v are real-valued, continuously differentiable, and satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

(d) f is continuous on Ω, and for every closed disc D contained inside Ω, we have the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}dw \quad \text{for all } z \text{ in the interior of } D;$$

(e) For every open disc D contained inside Ω, one can represent f as a convergent power series inside D, i.e. there exists coefficients $a_0, a_1, \dots,$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad \text{for all } z \in D,$$

where z_0 is the center of D;

(f) For every $z_0 \in \Omega$, there exists a non-empty open disc D centered at z_0, such that f can be represented as a convergent power series inside D;

(g) (Morera's theorem) f is continuous on Ω, and and for every closed triangle T contained inside Ω, we have

$$\int_{\partial T} f(z)dz = 0;$$

(h) For every open disc D contained inside Ω, there exists a holomorphic function F on D such that $F' = f$ on D.

Remarks.

1. The Cauchy-Riemann equations in (c) can be reformulated using the Cauchy-Riemann operator

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

See e.g. p.12 of [1].

2. The derivation of (e) from (d) can also be used to show that every function that is holomorphic on an annulus admits a Laurent series expansion. See e.g. Problem 3 of Chapter 3 of [1].

3. Other important consequences of the Cauchy integral formula in (d) include the Cauchy's inequalities, Liouville's theorem, and the fundamental theorem of algebra. See e.g. Section 4 of Chapter 2 of [1].

4. From the power series expansion of a holomorphic function into power series as in (e), one sees that the zeros of a non-constant holomorphic function are isolated. This leads one to study also singularities of holomorphic functions that are isolated, and those are classified into three kinds: removable singularities, poles and essential singularities. See e.g. Theorem 4.8 of Chapter 2, and Sections 1-3 of Chapter 3 of [1] (the latter also contains two important theorems, namely Riemann's removable singularity theorem, and the theorem of Casorati-Weierstrass on essential singularities).

5. It follows from Morera's theorem (g) that if $f_n \colon \Omega \to \mathbb{C}$ is a sequence of holomorphic functions, and f_n converges uniformly on every compact subset of Ω to a function $f \colon \Omega \to \mathbb{C}$, then f is also holomorphic on Ω. See e.g. Section 5.2 of Chapter 2 of [1].

6. Morera's theorem (g) has, as an important consequence, the symmetry principle for holomorphic functions. See e.g. Section 5.4 of Chapter 2 of [1].

7. One can also replace the open discs D in (h) above by any simply connected domain contained in Ω. In fact, given a holomorphic function f on a simply connected domain D, one can construct a desired F (known as a *primitive* of f) by
$$F(z) = \int_{\gamma_z} f(w)dw,$$
where γ_z is any (polygonal) path, entirely contained in D, that joins a fixed point $z_0 \in D$ to $z \in D$. In particular, if f is holomorphic and nowhere vanishing on a simply connected domain D, then one can construct a *logarithm* of f, by declaring it to be a primitive of the (holomorphic) function f'/f on D. See e.g. Sections 5 and 6 of Chapter 3 of [1].

8. The contour integral
$$\int_\gamma \frac{f'(z)}{f(z)} dz$$
also occurs in the discussion of the argument principle. If $f \colon \Omega \to \mathbb{C}$ is meromorphic on Ω, and γ is a positively oriented simple closed curve in Ω that avoids the poles and zeroes of f, then the above integral is $2\pi i$ times
$$(\text{number of zeroes of } f \text{ inside } \gamma - \text{ number of poles of } f \text{ inside } \gamma).$$
This in turn leads to three important theorems: Rouche's theorem, the open mapping theorem, and the maximum modulus principle. See e.g. Section 4 of Chapter 3 of [1].

References

[1] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003.