MATH4060 Exercise 6

Due Date: December 6, 2016.

The questions are from Stein and Shakarchi, *Complex Analysis*, unless otherwise stated.

Chapter 8. Exercise 11, 14.

Chapter 9. Exercise 2, 3, 4, 5, 6, 7.

Additional Exercises.
(Optional; you are not required to submit your solutions to the following questions.)

1. Let $\hat{\mathbb{C}}$ be the extended complex plane $\mathbb{C} \cup \{\infty\}$. A mapping $T\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form
$$T(z) = \frac{az+b}{cz+d}$$
where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ is called a Möbius transformation. (Here we interpret $T(\infty) = \infty$ if $c = 0$, and interpret $T(-d/c) = \infty$, $T(\infty) = a/c$ if $c \neq 0$.) Show that the set of all Möbius transformations form a group under composition; in particular, if
$$T(z) = \frac{az+b}{cz+d} \quad \text{and} \quad S(z) = \frac{a'z+b'}{c'z+d'},$$
then
$$(T \circ S)(z) = \frac{Az+B}{Cz+D} \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

2. A translation is a map of the form $z \mapsto z + b$ for some $b \in \mathbb{C}$. A (complex) dilation is a map of the form $z \mapsto az$ for some $a \in \mathbb{C} \setminus \{0\}$. The map $z \mapsto 1/z$ is called an inversion. Show that any Möbius transformation can be written as compositions of translations, (complex) dilations and inversions.

3. For any quadtuple $(z_1, z_2, z_3, z_4) \in \hat{\mathbb{C}}^4$, we define its cross ratio by
$$[z_1, z_2, z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \in \hat{\mathbb{C}}$$
(This is well-defined if z_1, z_2, z_3, z_4 are all distinct and in $\mathbb{C}$; we then extend by continuity to a continuous function from $\hat{\mathbb{C}}^4$ to $\hat{\mathbb{C}}$. It is called a cross ratio, because it can be written as $\frac{z_1-z_3}{z_2-z_3} : \frac{z_1-z_4}{z_2-z_4}$.)

   (a) Show that for any distinct $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation T such that $T(z_2) = 1$, $T(z_3) = 0$ and $T(z_4) = \infty$. Indeed, such T is given by the cross ratio
   $$T(z) = [z, z_2, z_3, z_4].$$

   (b) Show that Möbius transformations preserve the cross ratio; i.e.
   $$[S(z_1), S(z_2), S(z_3), S(z_4)] = [z_1, z_2, z_3, z_4]$$

for any Möbius transformation S and any $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. (Hint: Either use Question 2, or note that it suffices to show that for fixed distinct $z_2, z_3, z_4 \in \hat{\mathbb{C}}$, we have

$$[z, S(z_2), S(z_3), S(z_4)] = [S^{-1}(z), z_2, z_3, z_4] \quad \text{for all } z \in \hat{\mathbb{C}}.$$

This last identity follows from part (a), since the right hand side is a Möbius transformation that sends $S(z_2), S(z_3), S(z_4)$ to $1, 0, \infty$ respectively.)

4. A generalized circle in $\hat{\mathbb{C}}$ is either a straight line (including the point $\{\infty\}$), or a circle in $\mathbb{C}$.

   (a) Show that any Möbius transformation maps the real axis into a generalized circle. (Hint: This can be done by a direct computation. Let T be a Möbius transformation. If $w = T(x)$ for some $x \in \mathbb{R}$, then $T^{-1}(w) = \overline{T^{-1}(w)}$. Writing $T^{-1}(w) = \frac{aw+b}{cw+d}$ for some $a, b, c, d \in \mathbb{C}$ shows that w lies on a generalized circle.)

   (b) Suppose $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. Show that the four points lie on a generalized circle, if and only if $[z_1, z_2, z_3, z_4] \in \mathbb{R} \cup \{\infty\}$. (Hint: Without loss of generality assume that the four points z_1, z_2, z_3, z_4 are distinct. Now let

   $$T(z) := [z, z_2, z_3, z_4]$$

   be the Möbius transformation that maps z_2, z_3, z_4 to $1, 0, \infty$ respectively. Then

   $$[z_1, z_2, z_3, z_4] \in \mathbb{R} \cup \{\infty\} \quad \Leftrightarrow \quad T(z_1) \in \mathbb{R} \cup \{\infty\} \quad \Leftrightarrow \quad z_1 \in T^{-1}(\mathbb{R} \cup \{\infty\}),$$

   so one just needs to note that $T^{-1}(\mathbb{R} \cup \{\infty\})$ is the generalized circle that passes through z_2, z_3, z_4.)

   (c) Show that any Möbius transformation maps generalized circles to generalized circles. (Hint: Use part (b) and that Möbius transformations preserve cross ratios. Alternatively, one can use Question 2. Then one just needs to show, by direct computation, that each of the three basic kinds of Möbius transformations preserves generalized circles. The only difficult case is when $T(z) = 1/z$. But first let C be the circle in $\mathbb{C}$ given by the equation $|z - a|^2 = r^2$ for some $a \in \mathbb{C}$ and $r > 0$. Dividing by $|z|^2$, and expanding, this equation can be rewritten as

   $$(|a|^2 - r^2) \left| \frac{1}{z} \right|^2 - \frac{a}{z} - \frac{\bar{a}}{\bar{z}} + 1 = 0.$$

   Let $w = 1/z$. Depending on whether $r = |a|$ or not, this is the equation of a straight line or a circle in the w-plane. Similarly, if C is the straight line given by $bz + \bar{b}\bar{z} = c$, then the equation of C can be rewritten as

   $$\frac{c}{|z|^2} - \frac{b}{\bar{z}} - \frac{\bar{b}}{z} = 0,$$

   which is the equation of a circle or a straight line in the w-plane if $w = 1/z$.)