MATH4060 Exercise 5

Due Date: November 22, 2016.

The questions are from Stein and Shakarchi, *Complex Analysis*, unless otherwise stated.

Chapter 1. Exercise 7.

Chapter 2. Exercise 7.

Chapter 8. Exercise 1, 4, 5, 10, 12, 13.

Additional Exercises.

1. Find a biholomorphic map from the upper half disk $\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$ onto the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. (Hint: First show that the map $f(z) = \frac{1+z}{1-z}$ maps the upper half disk biholomorphically onto the first quadrant $\{x + iy : x > 0, y > 0\}$. You may express your answer as the composition of several simple maps.)

2. Find a biholomorphic map from the half-strip $\{z \in \mathbb{C} : -\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\}$ to the upper half space $\{w \in \mathbb{C} : \operatorname{Im} w > 0\}$. (Hint: Use Exercise 5 above.)

3. Let $\mathbb{D}$ be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. Suppose $f : \mathbb{D} \to \Omega$ is a biholomorphism from $\mathbb{D}$ onto a domain Ω. Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
be the power series expansion of f centered at 0. Show that the area of Ω is given by $\pi \sum_{n=1}^{\infty} n|a_n|^2$. (Hint: First show that the area of Ω is given by $\int_{\mathbb{D}} |f'(z)|^2 dxdy$.)

4. In this question we prove a special case of the so called three-lines lemma, which is useful for the next question.

   Let S be the vertical strip $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ for some $a, b \in \mathbb{R}$. Let $f : S \to \mathbb{C}$ be a holomorphic function on S, that extends continuously to the closure $\overline{S}$ of S. Suppose f is bounded on S (possibly by some very large constant M). If C is a constant for which $|f(z)| \leq C$ for all z on the boundary of S, show that $|f(z)| \leq C$ for all $z \in S$. (This is a generalization of the maximum modulus principle to an unbounded domain.)

   (Hint: Apply the maximum modulus principle to the function $e^{\varepsilon z^2} f(z)$ on $\overline{S}$ for $\varepsilon > 0$, and then let $\varepsilon \to 0^+$. The whole point here being that $|e^{\varepsilon z^2} f(z)| \to 0$ as $\operatorname{Im} z \to \pm\infty$, whenever $\varepsilon > 0$. You should check that this is indeed the case.)

5. For each $r > 1$, let A_r be the annuli $\{z \in \mathbb{C} : 1 < |z| < r\}$. The goal of this question is to show that if r_1, r_2 are both greater than 1, and $r_1 \neq r_2$, then there is no biholomorphic map from A_{r_1} onto A_{r_2}.

   Suppose $f : A_{r_1} \to A_{r_2}$ is a biholomorphic map for some $r_1, r_2 > 1$. We will show that $r_1 = r_2$.

   (a) Show that if $\delta > 0$ is sufficiently small, then
   $$\text{either} \quad f(A_{1+\delta}) \subset A_{\sqrt{r_1}}, \quad \text{or} \quad f(A_{1+\delta}) \subset A_{r_1} \setminus \overline{A_{\sqrt{r_1}}}.$$

In the latter case, by replacing f by r_2/f, we may reduce to the first case. Hence from now on, we assume that $f(A_{1+\delta}) \subset A_{\sqrt{r_1}}$ whenever δ is sufficiently small.

(b) Show (after the renormalization in part (a)) that

$$\lim_{\delta \to 0^+} \left(\max_{|z|=1+\delta} |f(z)| \right) = 1, \quad \text{and} \quad \lim_{\delta \to 0^+} \left(\min_{|z|=r_1-\delta} |f(z)| \right) = r_2.$$

(c) Write Log for the natural logarithm of the positive number. Show that the map $w \mapsto e^w$ maps the vertical strip $S := \{w \in \mathbb{C} \colon 0 < \operatorname{Re} w < \operatorname{Log} r_1\}$ into A_{r_1}.

(d) Part (c) allows us to define a holomorphic map $g \colon S \to A_{r_2}$, by

$$g(w) = f(e^w).$$

Let $\alpha = \dfrac{\operatorname{Log} r_2}{\operatorname{Log} r_1}$. Show that

$$|g(w)| = |e^{\alpha w}| \quad \text{for all } w \in S.$$

(Hint: Apply the three-lines lemma to the bounded holomorphic functions $g(w)/e^{\alpha w}$, and $e^{\alpha w}/g(w)$, on the slightly smaller vertical strip $\{w \in \mathbb{C} \colon \eta < \operatorname{Re} w < \operatorname{Log} r_1 - \eta\}$ than S, and let $\eta \to 0^+$.)

(e) Using part (d), show that there exists a constant c with $|c| = 1$ such that

$$f(e^w) = ce^{\alpha w} \quad \text{for all } w \in S.$$

(f) Show that α is an integer. (Hint: Replace w by $w + 2\pi i$ in the formula in part (e).)

(g) Conclude that

$$f(z) = cz^\alpha \quad \text{for all } z \in A_{r_1}.$$

Since f is injective, it follows that $\alpha = 1$, and hence $r_1 = r_2$.