Partial solution to midterm (revised)

1. (a) We claim that f is differentiable at a for arbitrary $a \in \mathbb{C}$. For $a \in \mathbb{C}$, there exists compact set K containing a. Since f_n converge to f uniformly on K, for any triangle $\triangle \subset K$,

$$\int_{\triangle} f_n \, dz \to \int_{\triangle} f \, dz.$$

But

$$\int_{\triangle} f_n \, dz = 0$$

as they are holomorphic. Thus by Moreras theorem, f is holomorphic on K because $\int_{\triangle} f \, dz = 0$. So, f is differentiable at a.

(b) Please refer to solution of HW2

2. Please refer to the tectbook.

3. (a) By cauchy formula,

$$f^{(n)}(a) = \frac{1}{2\pi i} \oint_{\partial B(a,1)} \frac{f(w)}{(w-a)^{n+1}} \, dw.$$

Thus,

$$\begin{aligned} |f^{(n)}(a)| &\leq \frac{1}{2\pi} \oint_{\partial B(a,1)} \frac{|f(w)|}{|w-a|^{n+1}} \, |dw| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + e^{i\theta}| \, d\theta \\ &\leq \frac{A}{2\pi} \int_0^{2\pi} \frac{1}{|1 + a + e^{i\theta}|^{2015}} \, d\theta \end{aligned}$$

If $|a| > 2$,

$$|1 + a + e^{i\theta}|^{2015} > C|a|^{2015}$$

where $C = 1/2^{2015}$. So,

$$\begin{aligned} |f^{(n)}(a)| &\leq \frac{A}{2\pi} \int_0^{2\pi} \frac{1}{|1 + a + e^{i\theta}|^{2015}} \, d\theta \\ &\leq \frac{A}{1 + C|a|^{2015}} \leq \frac{A}{C} \frac{1}{1 + |a|^{2015}}. \end{aligned}$$

If $|a| \leq 2$,

$$|f^{(n)}(a)| \leq A \leq \frac{A(1 + 2^{2015})}{1 + |a|^{2015}}.$$

Result follows when we choose $B = \sup\{\frac{A}{C}, A(1 + 2^{2015})\}$.

(b) Consider the function $g(z) = \overline{f(\bar{z})}$. Since

$$\frac{\partial}{\partial \bar{z}} \overline{f(\bar{z})} = \overline{\frac{\partial}{\partial z} f(\bar{z})} = 0,$$

g is holomorphic. You can also verify it by considering its power series expansion. Whenever $z = x \in [0,1]$, f is real. That is

$$f(x) = g(x) \ \ \forall x \in [0,1].$$

By identity theorem, $f(z) = g(z)$ for all $z \in \mathbb{C}$. Thus, $f(x)$ is real for any real x.

4. (a) Apply Weierstrass factorization theorem with $a_n = \log n$, and $p_n = n$. Noted that the choce of p_n is not unique.

(b) If there exists such function, let s be its order of growth. Then we will have for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{(\log n)^{s+\epsilon}} < +\infty.$$

But $\log n = O(n^{1/p})$ for any $p > 0$. By comparing it with harmonic series, it is impossible.

(c) Let $f(z)$ be an entire function which satisfies

$$f(\log n) = n \ \ \forall n \in \mathbb{N}.$$

Let $g(z) = f(z) - e^z$. Thus $g(\log n) = 0$ for all $n \in \mathbb{N}$ and g is of finite order of growth. If g is non-constant function, its zero set is discrete and hence countable. But as we observe in (b), it is not possible. Thus, $g \equiv 0$. Hence, $f(z) = e^z$.