MATH2040C Linear Algebra II
2017-18 Solution to Homework 6

Exercise 7.A

1 Let β be the standard basis for $\mathbb{F}^n$, which is orthonormal. By Theorem 7.10 in the book, we have $[T^*]_\beta = ([T]_\beta)^*$, so we see that

$$[T]_\beta = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \quad [T^*]_\beta = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Hence, the adjoint of T is given by $T^*(z_1, \dots, z_n) = (z_2, \dots, z_n, 0)$.

2 Consider the operator $T - \lambda I$ on V, by the last part of previous exercise, we see that

$$\dim \ker(T - \lambda I)^* = \dim \ker(T - \lambda I) + \dim V - \dim V = \dim \ker(T - \lambda I)$$

But $\ker(T - \lambda I)^* = \ker(T^* - \bar{\lambda} I^*) = \ker(T^* - \bar{\lambda} I)$. Hence, $\dim \ker(T - \lambda I) > 0$ if and only if $\dim \ker(T^* - \bar{\lambda} I) > 0$, which essentially means that λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.

3 Suppose U is T-invariant. For any $u^\perp \in U^\perp$, we have $\langle T^* u^\perp, u \rangle = \langle u^\perp, Tu \rangle = 0$ for any $u \in U$ as $Tu \in U$. This means $T^* u^\perp \in U^\perp$, i.e. $U^\perp$ is T^*-invariant.

4 First we recall a useful fact from Proposition 7.7 in the book

$$\operatorname{range} T^* = (\ker T)^\perp, \quad \ker T^* = (\operatorname{range} T)^\perp.$$

Since all spaces are finite dimensional, we have $U = \left(U^\perp\right)^\perp$ for any subspace U.

(a) From the above, we immediately see that T is injective if and only if $\ker T = \{0\}$, which means $\operatorname{range} T^* = V$ and T^* is surjective.

(b) Again, the result follows from taking orthogonal complement on the second equality.

5 (a) By Proposition 7.7 in the book, we have

$$\dim \ker T^* = \dim(\operatorname{range} T)^\perp = \dim W - \dim \operatorname{range} T.$$

Then by the fundamental theorem of linear maps

$$\dim V = \dim \ker T + \dim \operatorname{range} T,$$

we have

$$\dim \ker T^* = \dim W - \dim V + \dim \ker T.$$

(b) Using the fundamental theorem of linear maps with respect to T and T^*

$$\begin{aligned}\dim V &= \dim\ker T + \dim\operatorname{range} T;\\ \dim W &= \dim\ker T^* + \dim\operatorname{range} T^*\end{aligned}$$

and combining the result of (a), we can easily get

$$\dim\operatorname{range} T = \dim\operatorname{range} T^*.$$

6* (a) If T is self-adjoint, then for any polynomials $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$, we have that

$$\begin{aligned}& \langle Tp, q\rangle = \langle p, T^*q\rangle = \langle p, Tq\rangle\\ \Rightarrow\quad & \langle a_1x, b_0 + b_1x + b_2x^2\rangle = \langle a_0 + a_1x + a_2x^2, b_1x\rangle\\ \Rightarrow\quad & a_1\left(\frac{b_0}{2} + \frac{b_1}{3} + \frac{b_2}{4}\right) = b_1\left(\frac{a_0}{2} + \frac{a_1}{3} + \frac{a_2}{4}\right)\end{aligned}$$

This is obviously not true if we take $a_0 = a_2 = b_0 = b_1 = b_2 = 1$, and $a_1 = 0$.

(b) The reason is that $(1, x, x^2)$ is not an orthogonal basis.

14 We see that v and w are eigenvector, corresponding to distinct eigenvalues of T. By Proposition 7.22 in the book, v and w are orthogonal. Then, using Pythagoras theorem, we have

$$\begin{aligned}\|T(v+w)\|^2 &= \|3v + 4w\|^2\\ &= \|3v\|^2 + \|4w\|^2\\ &= 3^2 2^2 + 4^2 2^2 = 100.\end{aligned}$$

Hence, we have $\|T(v+w)\| = 10$.

15* First of all, we notice that, for any $v, w \in V$, we have

$$\langle v, T^*w\rangle = \langle Tv, w\rangle = \langle \langle v, u\rangle x, w\rangle = \langle v, u\rangle \langle x, w\rangle = \langle v, \langle w, x\rangle u\rangle.$$

So, $T^* \in \mathcal{L}(V)$ is given by $T^*w = \langle w, x\rangle u$.

(a) Suppose $\mathbb{F} = \mathbb{R}$. If $u = 0$, then the statement is trivial. So, let's assume $u \neq 0$. Now, we have

$$(T - T^*)v = \langle v, u\rangle x - \langle v, x\rangle u.$$

If T is self-adjoint, we simply take $v = u$ and as the left hand side vanishes, then we have $x = \frac{\langle u,x\rangle}{\langle u,u\rangle}u$. On the other hand, if $\{u, x\}$ is linearly dependent, say $x = \alpha u$, then $\langle v, u\rangle (\alpha u) - \langle v, \alpha u\rangle u$ is always zero in a real space, which means $(T - T^*)v = 0$ for any $v \in V$, that is, $T = T^*$.

(b) Again, we assume that $u \neq 0$. Consider the difference

$$\begin{aligned}(TT^* - T^*T)v &= \langle \langle v, x\rangle u, u\rangle x - \langle \langle v, u\rangle x, x\rangle u\\ &= \langle v, x\rangle \langle u, u\rangle x - \langle v, u\rangle \langle x, x\rangle u.\end{aligned}$$

Similarly, if T is normal, then we have a non-trivial sum of x and u. And if $x = \alpha u$ for some $\alpha \in \mathbb{F}$, we see that the right hand side vanished. Hence, the statement follows.

21 (a) For any $\alpha \in \mathbb{R}$ and any $f, g \in U_n$, say

$$f = a_0 + \sum a_k \cos kx + \sum b_k \sin kx,$$
$$g = c_0 + \sum c_k \cos kx + \sum d_k \sin kx,$$

we have

$$\begin{aligned} D(f+\alpha g) &= -\sum_{k=1}^{n} k(a_k + \alpha c_k)\sin kx + \sum_{k=1}^{n} k(b_k + \alpha d_k)\cos kx \qquad \in U_n \\ &= \left(-\sum_{k=1}^{n} k a_k \sin kx + \sum_{k=1}^{n} k b_k \cos kx\right) + \alpha\left(-\sum_{k=1}^{n} k c_k \sin kx + \sum_{k=1}^{n} k d_k \cos kx\right) \\ &= D(f) + \alpha D(g). \end{aligned}$$

This shows that D is a linear operator on U_n.
To see that $D^* = -D$, we first notice that $f(-\pi) = f(\pi)$ for any $f \in U_n$. Using integration by parts, we have, for any $f, g \in U_n$, that

$$\int_{-\pi}^{\pi} f'(x)g(x)\,dx = [f(x)g(x)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)g'(x)\,dx = \int_{-\pi}^{\pi} f(x)\left[-g'(x)\right]\,dx.$$

Hence, we have

$$\langle f, D^*(g)\rangle = \langle D(f), g\rangle = \langle f', g\rangle = \langle f, -g'\rangle = \langle f, -D(g)\rangle.$$

and $D^* = -D$. Obviously, D is normal, as $DD^* = -D^2 = D^*D$, but not self-adjoint.

(b) Again, we use integration by parts to get

$$\int_{-\pi}^{\pi} f''(x)g(x)\,dx = \left[f'(x)g(x)\right]_{-\pi}^{\pi} - \left[f(x)g'(x)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f(x)g''(x)\,dx = \int_{-\pi}^{\pi} f(x)g''(x)\,dx$$

for any $f, g \in U_n$ (so $f', g' \in U_n$ as D is an operator). This means

$$\langle f, T^*(g)\rangle = \langle T(f), g\rangle = \langle f'', g\rangle = \langle f, g''\rangle = \langle f, T(g)\rangle$$

for any $f, g \in U_n$, that is, $T^* = T$ is self-adjoint.

Exercise 7.B

2 It is equivalent to prove that $(T-2I)(T-3I) = 0$. By Theorem 7.29 in the book, there exists an orthogonal basis consisting of eigenvectors of T. So For any vector v, it can be decomposed as $v = v_1 + v_2$, where v_1, v_2 belong to the eigenspaces which can be spanned by the eigenvectors corresponding to the eigenvalue $2, 3$ respectively. Then we have $Tv_1 = 2v_1$ and $Tv_2 = 3v_2$, so

$$\begin{aligned} (T-2I)(T-3I)v &= (T-2I)(Tv_1 + Tv_2 - 3v_1 - 3v_2) \\ &= (T-2I)(-v_1) = -2v_1 + 2v_1 = 0 \end{aligned}$$

Hence $T^2 - 5T + 6I = 0$ holds.

3* Similar to the argument in Question 2 and combined with Proposition 7.24 in the book, if T is normal then $T^2 - 5T + 6I = 0$ also holds. So we need to construct an operator which is not normal. Similar to the example given on the first page of the lecture of week 12, we can define T as

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then T has only eigenvalues $2, 3$. And by simple calculation we have

$$T^2 - 5T + 6I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 6 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 15 & 5 \\ 0 & 0 & 15 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq 0.$$

4 By the complex Spectral theorem, we can substitute the statement " T is normal " by " there exists an orthogonal basis of V consisting of eigenvectors of T."

($\Rightarrow$) If there exists an orthogonal basis of V consisting of eigenvectors of T., then the results obviously holds.

($\Leftarrow$) To prove that there exists an orthogonal basis of V consisting of eigenvectors of T, it suffices to prove that in each eigenspace $E(\lambda_i, T)$, $1 \leq i \leq m$, we can find an orthogonal basis of the subspace $E(\lambda_i, T)$, while this can be done by the GramSchmidt Procedure.

6 ($\Rightarrow$) If T is self-adjoint, then by Theorem 7.13 in the book, its eigenvalues are all real.

($\Leftarrow$) If the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a normal operator T are all real, then we can take the corresponding eigenvectors $e_1, e_2, \cdots, e_n$ as an orthogonal basis of V. Then it suffices to prove that $Te_i = T^*e_i$, $\forall\ 1 \leq i \leq n$. While by Theorem 7.21 in the book, we have that

$$Te_i = \lambda_i e_i, \quad T^* e_i = \overline{\lambda_i} e_i,$$

then the result follows immediately if every λ_i is real.

7 If λ is an eigenvalue of T with an eigenvector v, then

$$Tv = \lambda v \Rightarrow (T^9 - T^8)v = (\lambda^9 - \lambda^8)v = 0 \Rightarrow \lambda^9 = \lambda^8$$

Thus the eigenvalues of T must be a root of $\lambda^9 = \lambda^8$, which implies that the eigenvalues of T can only be 0 or 1. Then by Problem 6, T is self-adjoint.
By the spectral theorem, T has a diagonal matrix M with respect to some orthonormal basis, where the diagonal entries can only be 0 or 1, so there holds $M^2 = M$, which implies $T^2 = T$.

9 By the complex spectral theorem, V has an orthogonal basis $e_1, \cdots, e_n$ which are eigenvalues of T corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ respectively. Then we choose $\mu_1, \cdots, \mu_n \in \mathbb{C}$, such that $\lambda_j = \mu_j^2$, $\forall\ j$. Define a linear operator $S \in \mathcal{L}(V)$ with

$$Se_j = \mu_j e_j, \quad \forall\ j$$

Then we have

$$S^2 e_j = \mu_j^2 e_j = \lambda_j e_j = Te_j, \quad \forall\ j$$

which implies $S^2 = T$.

Extra

1

$$p(\lambda) = \det(\lambda - A) = (\lambda-2)^3 - 3(\lambda-2) - 2$$

Note that $p(1) = 0$, then

$$\begin{aligned} p(\lambda) &= (\lambda-1-1)^3 - 3(\lambda-1) + 1 = (\lambda-1)^3 - 3(\lambda-1)^2 + 3(\lambda-1) - 1 - 3(\lambda-1) + 1 \\ &= (\lambda-1)^3 - 3(\lambda-1)^2 = (\lambda-1)^2(\lambda-4) \end{aligned}$$

Thus A has eigenvalues 1 and 4.

$\lambda = 1$:

$$(1-A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x+y+z \\ x-y-z \\ x-y-z \end{bmatrix} = 0 \Rightarrow \quad x = y+z$$

Then the eigenvectors of $\lambda = 1$ are $(x, y, z) = (1, 1, 0)$ and $(1, 0, 1)$. By the Gram-Schmidt Procedure on these two eigenvectors, we have two orthogonal eigenvectors

$$e_1 = \frac{1}{\sqrt{2}}(1, 1, 0), \quad e_2 = \frac{1}{\sqrt{6}}(1, -1, 2)$$

$\lambda = 4$:

$$(1-A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y+z \\ x+2y-z \\ x-y+2z \end{bmatrix} = 0 \Rightarrow \quad y = -x, z = -x.$$

Then the eigenvector of $\lambda = 4$ is $(x, y, z) = (-1, 1, 1)$, which can be normalized as

$$e_3 = \frac{1}{\sqrt{3}}(-1, 1, 1)$$

Thus the orthogonal matrix is

$$Q = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

2

$$p(\lambda) = \det(\lambda - A) = \lambda^3 - 3\lambda + 2$$

Note that $p(1) = 0$, then

$$\begin{aligned} p(\lambda) &= (\lambda-1+1)^3 - 3(\lambda-1) - 1 = (\lambda-1)^3 + 3(\lambda-1)^2 + 3(\lambda-1) + 1 - 3(\lambda-1) - 1 \\ &= (\lambda-1)^3 + 3(\lambda-1)^2 = (\lambda-1)^2(\lambda+2) \end{aligned}$$

Thus A has eigenvalues 1 and -2.

$\lambda = 1:$

$$(1-A)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & 1 & -1\\1 & 1 & -1\\-1 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x+y-z\\x+y-z\\-x-y+z\end{bmatrix} = 0 \Rightarrow \quad z = x+y$$

Then the eigenvectors of $\lambda = 1$ are $(x, y, z) = (1, 0, 1)$ and $(0, 1, 1)$. By the Gram-Schmidt Procedure on these two eigenvectors, we have two orthogonal eigenvectors

$$e_1 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad e_2 = \frac{1}{\sqrt{6}}(1, 2, -1)$$

$\lambda = -2:$

$$(-2-A)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2 & 1 & -1\\1 & -2 & -1\\-1 & -1 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2x+y-z\\x-2y-z\\-x-y-2z\end{bmatrix} = 0 \Rightarrow \quad y = x, z = -x.$$

Then the eigenvector of $\lambda = 4$ is $(x, y, z) = (1, 1, -1)$, which can be normalized as

$$e_3 = \frac{1}{\sqrt{3}}(1, 1, -1)$$

Thus the orthogonal matrix is

$$Q = \begin{pmatrix}e_1 & e_2 & e_3\end{pmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\end{bmatrix}$$