MATH2040C Linear Algebra II
2017-18 Solution to Homework 3

Exercise 3.A

1* First we prove the "if" part. $b = c = 0$ implies that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined as $T(x, y, z) = (2x - 4y + 3z, 6x)$. For any $(x, y, z), (u, v, w) \in \mathbb{R}^3$,

$$\begin{aligned} T((x,y,z) + (u,v,w)) &= T(x+u, y+v, z+w) \\ &= (2(x+u) - 4(y+v) + 3(z+w), 6(x+u)) \\ &= (2x - 4y + 3z, 6x) + (2u - 4v + 3w, 6u) \\ &= T(x,y,z) + T(u,v,w) \end{aligned}$$

Thus, this map is additive.

For any $(x, y, z) \in \mathbb{R}^3$ and $a \in \mathbb{R}$,

$$\begin{aligned} T(a(x,y,z)) &= T(ax, ay, az) \\ &= (2ax - 4ay + 3az, 6ax) \\ &= a(2x - 4y + 2z, 6x) \\ &= aT(x,y,z) \end{aligned}$$

Thus, this map is homogeneous of degree 1. We conclude that T is a linear map.

Then we prove the "only if" part. Now T is linear, thus additive and homogeneous of degree 1 by definition. For any $(x, y, z) \in \mathbb{R}^3$ and $a \in \mathbb{R}$,

$$\begin{aligned} T(a(x,y,z)) &= T(ax, ay, az) \\ &= (2ax - 4ay + 3az + b, 6ax + ca^3xyz) \end{aligned}$$

and

$$\begin{aligned} aT(x,y,z) &= a(2x - 4y + 2z + b, 6x + cxyz) \\ &= (2ax - 4ay + 3az + ab, 6ax + caxyz) \end{aligned}$$

By homogeneity, $T(a(x, y, z)) = aT(x, y, z)$. This implies that $(2ax - 4ay + 3az + b, 6ax + ca^3xyz) = (2ax - 4ay + 3az + ab, 6ax + caxyz)$ for any $(x, y, z) \in \mathbb{R}^3$ and $a \in \mathbb{R}$. Thus $b = ab$, $ca^3xyz = caxyz$ for any $x, y, z, a \in \mathbb{R}$. This can only happen when $b = c = 0$ (say, take $a = 2$ and $x = y = z = 1$ to see this). This proves the "only if" part.

4* Suppose there exists $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$a_1v_1 + \cdots a_mv_m = \mathbf{0}.$$

Then apply the linear transformation T on both sides, we have

$$T(a_1v_1 + \cdots a_mv_m) = T(\mathbf{0}).$$

By linearity of T, we have

$$a_1Tv_1 + \cdots a_mTv_m = \mathbf{0}.$$

Since $(Tv_1, \ldots, Tv_m)$ is given to be linearly independent, by definition, we have $a_1, \ldots, a_m$ being all zero. Therefore $(v_1, \ldots, v_m)$ is linearly independent.

9 Consider the conjugation function on $\mathbb{C}$ defined by $\varphi(a+bi) = a - bi$ for all $a, b \in \mathbb{R}$. Let $w = a + bi, z = c + di \in \mathbb{C}$ with $a, b, c, d \in \mathbb{R}$. We see that

$$\varphi(w) + \varphi(z) = (a - bi) + (c - di) = (a + c) - (b + d)i = \varphi((a + c) + (b + d)i) = \varphi(w + z).$$

However, we check that $i\varphi(1) = i \cdot 1 = i$ while $\varphi(i \cdot 1) = \varphi(i) = -i \neq i$. Therefore φ is not $\mathbb{C}$-linear.

(For the $\mathbb{R}$ case: Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Let $(1, \pi)$ be a list of vectors in $\mathbb{R}$. It is linearly independent over $\mathbb{Q}$ by irrationality of π. The advance tools that we will use are "Every linearly independent subset of a vector space can be extended to a basis" and "Value at basis of domain determines a linear map" which are analogs to Theorem 2.33 and 3.5 of textbook in vector spaces which are not necessarily finite dimensional. Let β be a $\mathbb{Q}$-basis of $\mathbb{R}$ containing 1 and π. Define a $\mathbb{Q}$-linear operator φ on $\mathbb{R}$ such that $\varphi(1) = \pi$, $\varphi(\pi) = 1$, and $\varphi(x) = x$ for $x \in \beta$ not equal to 1 nor π. Then it is additive but not $\mathbb{R}$-linear since $\pi\varphi(1) = \pi^2$ while $\varphi(\pi) = 1$.)

11 V is a finite dimensional vector space and U is a subspace of V. Hence we can pick a basis $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k\}$ of U, which extends to a basis $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_l\}$ of V. $S(\boldsymbol{u}_1), \cdots, S(\boldsymbol{u}_k)$ are vectors in W and we pick ℓ vectors $\{\boldsymbol{w}_1, \cdots, \boldsymbol{w}_\ell\}$ in W. Then by Theorem 3.5 in the textbook, we have a unique linear map $T : V \to W$ such that $T\boldsymbol{u}_i = S\boldsymbol{u}_{\boldsymbol{i}}$ for $i = 1, \cdots, k$ and $T\boldsymbol{v}_j = \boldsymbol{w}_j$ for $j = 1, \cdots, \ell$.

Finally, $T\boldsymbol{u} = S\boldsymbol{u}$ for all $\boldsymbol{u} \in U$. Indeed, for any $\boldsymbol{u} \in U$, write $\boldsymbol{u} = a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k$ since $\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_k\}$ is a basis of U. Then

$$\begin{aligned} T\boldsymbol{u} &= T(a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k) \\ &= a_1T\boldsymbol{u}_1 + \cdots + a_kT\boldsymbol{u}_k \\ &= a_1S\boldsymbol{u}_1 + \cdots + a_kS\boldsymbol{u}_k \\ &= S(a_1\boldsymbol{u}_1 + \cdots + a_k\boldsymbol{u}_k) \\ &= S\boldsymbol{u} \end{aligned}$$

Note that we used above the linearity of S and T.

Exercise 3.B

5* Let $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_4\}$ be the standard basis of $\mathbb{R}^4$. Let $T\boldsymbol{v}_1 = T\boldsymbol{v}_2 = 0$, $T\boldsymbol{v}_3 = \boldsymbol{v}_1$ and $T\boldsymbol{v}_4 = \boldsymbol{v}_2$. For any $\boldsymbol{v} \in \mathbb{R}^4$, $\boldsymbol{v} = a_1\boldsymbol{v}_1 + \cdots + a_4\boldsymbol{v}_4$ for some $a_1, \cdots, a_4$. Define $T\boldsymbol{v} = a_1T\boldsymbol{v}_1 + \cdots + a_4T\boldsymbol{v}_4$. This defines a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$. Then $\ker T = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \operatorname{range} T$ (Here we used Exercise 3.B Q10).

6* Suppose that there exists such a linear map. By the fundamental theorem $\dim \operatorname{range} T + \dim \ker T = 5$. And by assumption, $\ker T = \operatorname{range} T$. Thus $\dim \operatorname{range} T = \dim \ker T = 2.5$. This is absurd because by definition, dimensions are integers. This shows that there does not exist such a linear map.

8* Let $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ be bases of V and W respectively, where $n = \dim V$ and $m = \dim W$. It is given that $n \geq m \geq 2$. Define linear maps $T, S \in \mathcal{L}(V, W)$ by

$$T(v_i) = \begin{cases} w_i & \text{for } i = 1, \ldots, m-1; \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

and

$$S(v_i) = \begin{cases} w_m & \text{for } i = m; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

We claim that T and S are not surjective. Note that

$$\operatorname{range} T = \operatorname{span}\{T(v_1), \ldots, T(v_n)\} = \operatorname{span}\{w_1, \ldots, w_{m-1}, \mathbf{0}, \ldots, \mathbf{0}\}$$

and

$$\operatorname{range} S = \operatorname{span}\{S(v_1), \ldots, S(v_n)\} = \operatorname{span}\{\mathbf{0}, \ldots, \mathbf{0}, w_m, \mathbf{0}, \ldots, \mathbf{0}\}$$

(Here we used Exercise 3.B Q10). Since $(w_1, \ldots, w_m)$ is linearly independent by construction, $w_m \notin \{w_1, \ldots, w_{m-1}\} = \operatorname{range} T$ and $w_1 \notin \operatorname{span}\{w_m\} = \operatorname{range} S$.

Now the sum $(T + S)$ satisfies

$$(T + S)(v_i) = \begin{cases} w_i & \text{for } i = 1, \ldots, m; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Therefore $\operatorname{range}(T+S) = \operatorname{span}\{(T+S)(v_1), \ldots, (T+S)(v_n)\} = \operatorname{span}\{w_1, \ldots, w_m, \mathbf{0}, \ldots, \mathbf{0}\} = \operatorname{span}\{w_1, \ldots, w_m\} = W$. Therefore the sum of two non-surjective map can be surjective and the set

$$\{T \in \mathcal{L}(V, W) : T \text{ is not surjective.}\}$$

is not a subspace of $\mathcal{L}(V, W)$.

9 Suppose there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$a_1 T v_1 + \cdots a_n T v_n = \mathbf{0}.$$

Since T is linear, we have

$$T(a_1 v_1 + \cdots a_n v_n) = \mathbf{0}.$$

By injectivity of T, we have

$$a_1 v_1 + \cdots a_n v_n = \mathbf{0}.$$

Since $(v_1, \ldots, v_n)$ is given to be linearly independent, by definition, we have $a_1, \ldots, a_n$ being all zero. Therefore $(T v_1, \ldots, T v_n)$ is linearly independent in W.

10 By definition of range, $T v_1, \ldots, T v_n \in \operatorname{range} T$. Therefore $\operatorname{span}(T v_1 \ldots, T v_n) \subset \operatorname{range} T$.

Let $w \in \operatorname{range} T$. There exists $v \in V$ such that $T(v) = w$. Since $v_1, \ldots, v_n$ spans V, there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that $a_1 v_1 + \cdots + a_n v_n = v$. Thus

$$w = T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T v_1 + \cdots + a_n T v_n \in \operatorname{span}(T v_1 \ldots, T v_n).$$

Hence $\operatorname{span}(T v_1 \ldots, T v_n) = \operatorname{range} T$.

12* Since V is finite dimensional, $\operatorname{null} T$ is finite dimensional too. Let $(v_1, \ldots, v_n)$ be a basis of $\operatorname{null} T$. Extend it to $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ a basis of V. We claim that $U = \operatorname{span}(w_1, \ldots, w_m)$ has the desired property. By construction, it is a subspace of V. By Theorem 2.34 in the textbook, we have $U \oplus \operatorname{null} T = V$. In particular, $U \cap \operatorname{null} T = \{\mathbf{0}\}$. By definition of range, $Tu \in \operatorname{range} T$ for all u in U. Therefore $\{Tu : u \in U\} \subset \operatorname{range} T$. It remains to show that $\operatorname{range} T \subset \{Tu : u \in U\}$. Suppose $w \in \operatorname{range} T$. Then there exists $v \in V$ such that $w = T(v)$. Since $U + \operatorname{null} T = V$, there exists $u \in U$, $x \in \operatorname{null} T$ such that $v = u + x$. Therefore

$$w = T(v) = T(u + x) = Tu + Tx = Tu + \mathbf{0} = Tu$$

and $\operatorname{range} T = \{Tu : u \in U\}$.

15 Let $N = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$. We claim that $\dim N = 2$. It suffices to show that $\operatorname{span}\{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\} = N$ since $\{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$ is linearly independent (compare the components). By direct check $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1) \in N$ so $\operatorname{span}\{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\} \subset N$. If $x = (x_1, x_2, x_3, x_4, x_5) \in N$, $x_1 = 3x_2$ and $x_3 = x_4 = x_5$. Therefore $x = x_2(3, 1, 0, 0, 0) + x_5(0, 0, 1, 1, 1) \in \operatorname{span}\{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}$. Hence we have the claim.
Since $\dim N = 2$, if N was the kernel of a linear map T from $\mathbb{F}^5$ to $\mathbb{F}^2$, then by the fundamental theorem, we would have

$$\dim \operatorname{null} T + \dim \operatorname{range} T = 5.$$

However, $\operatorname{range} T \subset \mathbb{F}^2$ so $\dim \operatorname{range} T \leq 2$. Therefore $5 = \dim \operatorname{null} T + \dim \operatorname{range} T \leq 2 + 2 = 4$, a contradiction. Therefore there is no such transformation.

18 Let $\{v_1, \ldots, v_n\}$ be a basis of V.

($\Rightarrow$) Suppose T is a surjective linear map from V onto W. Then $\operatorname{range} T = W$. By Exercise 3.B Q10, $\operatorname{range} T = \operatorname{span}(Tv_1 \ldots, Tv_n)$. Therefore W is span by n vectors. Since length of a spanning list is not less than the dimension, $\dim W \leq n = \dim V$.

($\Leftarrow$) Suppose $m := \dim W \leq \dim V$, let $\{w_1, \ldots, w_m\}$ be a basis of W. Define a linear map T from V to W by

$$T(v_i) = \begin{cases} w_i & \text{for } i = 1, \ldots, m; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

This is possible since $n \geq m$. Then we have $\operatorname{range} T = \operatorname{span}\{T(v_1), \ldots, T(v_n)\} = \operatorname{span}\{w_1, \ldots, w_m, \mathbf{0}, \ldots, \mathbf{0}\} = \operatorname{span}\{w_1, \ldots, w_m\} = W$ and T is surjective.

22 Since U is finite dimensional, let $\alpha = \{u_1, \ldots, u_n\}$ be a basis of $\operatorname{null} T$. If $v \in \operatorname{null} T$, $ST(v) = S(\mathbf{0}) = \mathbf{0}$. Therefore $v \in \operatorname{null} ST$ and $\operatorname{null} T \subset \operatorname{null} ST$. Extend α to a basis $\beta = \{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ of $\operatorname{null} ST$. Note that $ST(v_i) = \mathbf{0}$ for all i. Therefore $\{Tv_1, \ldots Tv_m\} \subset \operatorname{null} S$. We claim that $\{Tv_1, \ldots Tv_m\}$ is linearly independent in $\operatorname{null} S$. Suppose there exists $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$a_1Tv_1 + \cdots a_mTv_m = \mathbf{0}.$$

Since T is linear, we have

$$T(a_1v_1 + \cdots a_mv_m) = \mathbf{0}.$$

Therefore $a_1v_1 + \cdots a_mv_m \in \operatorname{null} T$ and there exists $b_1, \ldots, b_n \in \mathbb{F}$ such that

$$a_1v_1 + \cdots a_nv_m = b_1u_1 + \cdots + b_nu_n.$$

Rewriting, we have

$$a_1v_1 + \cdots a_nv_m - b_1u_1 - \cdots - b_nu_n = \mathbf{0}.$$

Since $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ is constructed to be linearly independent, by definition, we have $a_1, \ldots, a_m, b_1, \ldots, b_n$ being all zero. Therefore $\{Tv_1, \ldots, Tv_m\}$ is linearly independent in null S. Hence $\dim \operatorname{null} S \geq \#\{Tv_1, \ldots, Tv_m\} = m$. Therefore

$$\dim \operatorname{null} ST = n + m \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

27 Suppose $p \in \mathcal{P}(\mathbb{R})$. If $p \equiv 0$ the zero polynomial, take $q \equiv 0 \in \mathcal{P}(\mathbb{R})$. So we may assume $p \not\equiv 0$. Let d be the degree of p which is a non-negative integer. Let $V = \mathcal{P}_{d+1}(\mathbb{R})$ and $W = \mathcal{P}_d(\mathbb{R})$. Define the linear map $T : V \to W$ by $T(f) = 5f'' + 3f'$. Suppose $f \in V$ such that $T(f) \equiv 0$, i.e. $5f'' + 3f' = 0$. Then $5f' + 3f \equiv c$ where c is a constant. If $f \not\equiv 0$, then the degree of f is greater than the degree of f'. So the highest degree term of $3f + 5f'$ is that of $3f$. By comparing the coefficient of the highest degree term on both sides, the degree of f can only be 0 and so $f' \equiv 0$. Hence f can only be constant. It is also true that $Tf \equiv 0$ for any constant polynomial f. Therefore the kernel of T must be the subspace of constant functions which has dimension 1. By fundamental theorem, $\dim \operatorname{null} T + \dim \operatorname{range} T = \dim V = d + 2$. Therefore $\dim \operatorname{range} T = d+2-1 = d+1 = \dim W$ and $\operatorname{range} T = W$ (note that $\operatorname{range} T \subset W$ by construction of T). So T is surjective and there exists $q \in \mathcal{P}_{d+1}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ such that $Tq = 5q'' + 3q' = p$.

Exercise 3.C

2 Let $\beta = (x^3, x^2, x, 1) \subset \mathcal{P}_3(\mathbb{R})$ and $\gamma = (3x^2, 2x, 1) \subset \mathcal{P}_2(\mathbb{R})$. Since the elements in β (resp. γ) have different degree, they are linearly independent. Since $|\beta| = \dim \mathcal{P}_3(\mathbb{R})$ and $|\gamma| = \dim \mathcal{P}_2(\mathbb{R})$, they are basis of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ respectively.

Now we have $Dx^3 = 3x^2$, $Dx^2 = 2x$, $Dx = 1$, $D1 = 0$. Therefore we have

$$\begin{aligned}\mathcal{M}(D, \beta, \gamma) &= \begin{bmatrix}\mathcal{M}(Dx^3, \gamma) & \mathcal{M}(Dx^2, \gamma) & \mathcal{M}(Dx, \gamma) & \mathcal{M}(D1, \gamma)\end{bmatrix} \\ &= \begin{bmatrix}\mathcal{M}(3x^2, \gamma) & \mathcal{M}(2x, \gamma) & \mathcal{M}(1, \gamma) & \mathcal{M}(0, \gamma)\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0\end{bmatrix}\end{aligned}$$

3 Let $(u_1, \ldots, u_r)$ be a basis of null T. Extends it to a basis $(u_1, \ldots, u_r, v_1, \ldots, v_n)$ of V. We may we order it into $\alpha := (v_1, \ldots, v_n, u_1, \ldots, u_r)$.

We claim that $(Tv_1, \ldots Tv_n)$ is linearly independent in W. Suppose there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$a_1Tv_1 + \cdots a_nTv_n = \mathbf{0}.$$

Since T is linear, we have

$$T(a_1v_1 + \cdots a_nv_n) = \mathbf{0}.$$

Therefore $a_1v_1 + \cdots a_nv_n \in \operatorname{null} T$ and there exists $b_1, \ldots, b_r \in \mathbb{F}$ such that

$$a_1v_1 + \cdots a_nv_n = b_1u_1 + \cdots + b_ru_r.$$

Rewriting, we have

$$a_1v_1 + \cdots a_nv_n - b_1u_1 - \cdots - b_ru_r = \mathbf{0}.$$

Since $(v_1, \dots, v_n, u_1, \dots, u_r)$ is constructed to be linearly independent, by definition, we have $a_1, \dots, a_n, b_1, \dots, b_r$ being all zero. Therefore $(Tv_1, \dots, Tv_n)$ is linearly independent in W.

Extend $(Tv_1, \dots, Tv_n)$ to a basis $\beta := (Tv_1, \dots, Tv_n, w_1, \dots, w_m)$ of W. We check that

$$\mathcal{M}(Tv_i, \beta) = e_i,\ \mathcal{M}(Tu_j, \beta) = \mathbf{0} \text{ for } i = 1, \dots, n,\ j = 1, \dots r$$

where e_i is the $(n+m) \times 1$ column vector with only non-zero entry is the i-th one with value 1. Hence

$$\mathcal{M}(T, \alpha, \beta) = \begin{bmatrix} e_1 & e_2 & \cdots & e_n & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

By Exercise 3.B Q10, we have

$$\operatorname{range} T = \operatorname{span}(Tv_1, \dots, Tv_n, Tu_1, \dots, Tu_r) = \operatorname{span}(Tv_1, \dots, Tv_n).$$

Since $(Tv_1, \dots, Tv_n)$ is linearly independent, $\dim \operatorname{range} T = n$.

Exercise 3.D

1 Note that

$$(ST)T^{-1}S^{-1} = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$$

and

$$T^{-1}S^{-1}(ST) = T(SS^{-1})T^{-1} = TIT^{-1} = TT^{-1} = I.$$

Therefore ST is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

2 Let $(v_1, \dots, v_n)$ be a basis of V where $n = \dim V \geq 2$. Define linear maps $T, S \in \mathcal{L}(V)$ by

$$T(v_i) = \begin{cases} v_i & \text{for } i = 1, \dots, n-1; \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

and

$$S(v_i) = \begin{cases} v_n & \text{for } i = n; \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

We claim that T and S are non-invertible. Note that $T(v_n) = \mathbf{0}$ and $S(v_1) = \mathbf{0}$. Therefore they are not injective and hence non-invertible.

Now the sum $(T + S)$ satisfies

$$(T + S)(v_i) = v_i \text{ for all } i.$$

Therefore $(T + S) = I_V$ and hence invertible. As a result,

$$\{T \in \mathcal{L}(V) : T \text{ is not invertible.}\}$$

is not a subspace of $\mathcal{L}(V)$.

4 ($\Rightarrow$) Assume $\operatorname{null} T_1 = \operatorname{null} T_2$. Let $U_1 = \operatorname{range} T_1$, $U_2 = \operatorname{range} T_2$. We claim that U_1 and U_2 are isomorphic.

For $u \in U_2$, we claim that for any $v, v' \in V$ satisfying $T_2 v = T_2 v' = u$, we have $T_1 v = T_1 v'$. Indeed, if $T_2 v = T_2 v'$ then $T_2(v - v') = T_2 v - T_2 v' = u - u = \mathbf{0}$. By assumption, $v - v' \in \operatorname{null} T_1$. Therefore $T_1 v - T_1 v' = T_1(v - v') = \mathbf{0}$. Hence we may define a function $Q : U_2 \to U_1$ by $Qu = T_1 v$ for any $u \in U_2$ and $v \in V$ such that $u = T_2 v$. Suppose $u, u' \in U_2$ and $v, v' \in V$ such that $u = T_2 v$ and $u' = T_2 v'$ and $\lambda \in \mathbb{F}$, we have $\lambda u + u' = \lambda T_2 v + T_2 v' = T_2(\lambda v + v')$ and $\lambda v + v' \in V$. Therefore $Q(\lambda u + u') = T_1(\lambda v + v') = \lambda T_1 v + T_1 v' = \lambda Qu + Qu'$. Hence Q is linear. Suppose $u \in U_2$ such that $Qu = \mathbf{0}$. Pick $v \in V$ such that $T_2 v = u$. Then $Qu = \mathbf{0}$ implies $T_1 v = \mathbf{0}$ and $v \in \operatorname{null} T_1 = \operatorname{null} T_2$. Thus $u = \mathbf{0}$ and Q is injective. Let $u' \in U_1$. Pick $v \in V$ such that $T_1 v = u'$. Let $u = T_2 v$. Then $Qu = T_1 v = u'$ and Q is surjective. Hence Q is bijective and invertible.

Now we have U_1 and U_2 being isomorphic through the isomorphism Q. In particular, $\dim U_1 = \dim U_2$. By Theorem 2.34 in the textbook, there exist subspaces Z_1, Z_2 of W such that $W = Z_1 \oplus U_1 = Z_2 \oplus U_2$. Apply Theorem 2.43 in the textbook, $\dim W = \dim Z_1 + \dim U_1 = \dim Z_2 + \dim U_2$. So $\dim Z_1 = \dim Z_2$ since every terms in the previous equation are just integers. By Theorem 3.59 in the textbook, Z_1 is isomorphic to Z_2. Let $R : Z_2 \to Z_1$ be such an isomorphism. For any $w \in W$, since we have $W = Z_2 \oplus U_2$, there exist unique $z \in Z_2$ and $u \in U_2$ such that $w = z + u$. Define a function $S : W \to W$ by $Sw = Rz + Qu$. Suppose $\lambda \in \mathbb{F}$ and $w' = z' + u' \in W$ such that $z' \in Z_2$ and $u' \in U_2$. Then $\lambda w + w' = \lambda(z + u) + z' + u' = (\lambda z + z') + (\lambda u + u')$. Note that $\lambda z + z' \in Z_2$ and $\lambda u + u' \in U_2$ since they are vector subspaces. Therefore this is the decomposition of $\lambda w + w'$ into a sum of an element of Z_2 and an element of U_2. Hence

$$\begin{aligned} S(\lambda w + w') &= R(\lambda z + z') + Q(\lambda u + u') = \lambda Rz + Rz' + \lambda Qu + Qu' \\ &= \lambda(Rz + Qu) + (Rz' + Qu') = \lambda Sw + Sw' \end{aligned}$$

and so S is linear. If $w = z + u \in W$ such that $z \in Z_2$ and $u \in U_2$ and $Sw = \mathbf{0}$. Then $Rz + Qu = \mathbf{0}$. Since $Rz \in Z_1$, $Qu \in U_1$ and $Z_1 \cap U_1 = \{\mathbf{0}\}$, $Rz = Qu = \mathbf{0}$. Since R and Q are invertible, they are injective. Therefore $z = u = \mathbf{0}$. As a result S is injective. By Theorem 3.69 of textbook, S is invertible.

($\Leftarrow$) Assume there exists invertible $S \in \mathcal{L}(V)$ such that $ST_2 = T_1$. Suppose $v \in \operatorname{null} T_1$ then $\mathbf{0} = T_1 v = ST_2 v$. Since S is invertible, it is injective. Therefore $T_2 v = \mathbf{0}$ and $v \in \operatorname{null} T_2$. Suppose $u \in \operatorname{null} T_2$. Then $T_1 u = ST_2 u = S\mathbf{0} = \mathbf{0}$. Hence $u \in \operatorname{null} T_1$. Hence $\operatorname{null} T_1 = \operatorname{null} T_2$.

(Remark: the "only if" direction will be much easier if we have the quotient space construction. By Theorem 3.91, we have $\operatorname{range} T_1$ isomorphic to $V/\operatorname{null} T_1$ which is equal to $V/\operatorname{null} T_2$, which in turn isomorphic to $\operatorname{range} T_2$. Therefore the isomorphism Q can be obtained easily.)

7* (a) We have $T_0 \in E$ since $T_0 v = \mathbf{0}$ by definition of T_0. Suppose $T, S \in E$, i.e. $Tv = 0$ and $Sv = 0$. Then $(T + S)v = Tv + Sv = 0 + 0 = 0$ and $(aT)v = a(Tv) = a(0) = 0$ for any $a \in \mathbb{F}$. Hence, E is closed under addition and scalar multiplication, which means E is a subspace.

(b) Let $\dim V = n$ and $\dim W = m$. If $v \neq \mathbf{0}$, let $U = \operatorname{span}\{v\}$. We can write $V = U \oplus V'$ for some subspace V' of V with $\dim V' = n - 1$. We claim that $\mathcal{L}(V', W)$ is isomorphic to E. For $x \in V$, there exists unique $u \in \operatorname{span}\{v\}$ and $z \in V'$ such that $x = u + z$. Define a function $(\bullet)_! : \mathcal{L}(V', W) \to E$ by $T_!(x) = Tz$. We check that $T_! \in E$. Note that for $\lambda \in \mathbb{F}$ and $x' \in V$, there exists unique $u' \in \operatorname{span}\{v\}$ and $z' \in V'$ such that $x' = u' + z'$. So $\lambda x + x' = (\lambda u + u') + (\lambda z + z')$ is the unique decomposition. Now $T_!(\lambda x + x') = T(\lambda z + z') = \lambda Tz + Tz' = \lambda T_!(x) + T_!(x')$. Also for $u \in U$, $u = u + \mathbf{0}$. Therefore $T_!(u) = T\mathbf{0} = \mathbf{0}$. Now we check that $(\bullet)_!$ is linear. We have

$$(\lambda T + S)_!(x) = (\lambda T + S)(z) = \lambda Tz + Sz = \lambda(T_! x) + S_! x.$$

Let $(\bullet)|_{V'}$ be the restriction map from $E \subset \mathcal{L}(V, W)$ to $\mathcal{L}(V', W)$. It is easy to check that it is linear and the composition $(\bullet)|_{V'} \circ (\bullet)_!$ is the identity map on $\mathcal{L}(V', W)$.
Now we check that the composition $(\bullet)_! \circ (\bullet)|_{V'}$ is the identity map on E. Let $T \in E$. For all $x \in V$ with decomposition $x = u + z$, we have

$$(T|_{V'})_!(x) = (T|_{V'})(z) = Tz = Tu + Tz = Tx.$$

Since x is arbitrary, $(T|_{V'})_! = T$. Therefore we obtain an isomorphism between E and $\mathcal{L}(V', W)$. Hence we have the following formula

$$\dim E = \dim\{T \in \mathcal{L}(V, W) : Tv = 0\} = \dim \mathcal{L}(V', W) = \dim V' \dim W = (n-1)m.$$

10* Suppose $ST = I$. Assume $v \in V$ such that $Tv = \mathbf{0}$. Then $v = Iv = STv = S\mathbf{0} = \mathbf{0}$ and T is injective. By Theorem 3.69 in the textbook, T is invertible. So there exists $T^{-1} \in \mathcal{L}(V)$ such that $TT^{-1} = I$. In particular

$$S = SI = STT^{-1} = IT^{-1} = T^{-1}.$$

By definition of inverse, $TS = TT^{-1} = I$. The reverse direction can be achieved by exchanging T and S in the above proof.

18 Define a map $\operatorname{eval}_1 : \mathcal{L}(\mathbb{F}, V) \to V$ by $(\operatorname{eval}_1 T) = T(1)$ for all $T \in \mathcal{L}(\mathbb{F}, V)$. Note that for all $T, S \in \mathcal{L}(\mathbb{F}, V)$, $\lambda \in \mathbb{F}$, we have

$$\operatorname{eval}_1(\lambda T + S) = (\lambda T + S)(1) = \lambda T(1) + S(1) = \lambda \operatorname{eval}_1(T) + \operatorname{eval}_1(S)$$

and hence eval_1 is linear. If $\operatorname{eval}_1(T) = \mathbf{0}$ the zero vector, for all $\lambda \in \mathbb{F}$, $T(\lambda) = \lambda T(1) = \lambda\mathbf{0} = \mathbf{0}$. Therefore $T = T_0$ the zero transformation and eval_1 is injective. For $v \in V$, define $T_v : \mathbb{F} \to V$ by $T_v(c) = cv$. It is clearly linear and $\operatorname{eval}_1(T_v) = T_v(1) = 1v = v$. Therefore eval_1 is surjective. Hence eval_1 is invertible and thus V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic.