MATH2040C Linear Algebra II
2017-18 Solutions to Homework 1

Exercise 1.B

4 The empty set only fails to satisfy the existence of additive identity (VS3). Since empty set contains no elements, there does not exists an element $\overrightarrow{0}$ in it. For others, they require all elements of the empty set having some certain properties. These are examples of vacuously true statements.

6 $\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$ is not a vector space over $\mathbb{R}$. It fails to satisfy associativity of addition (VS2) and distributivity (VS7ii). For (VS2), note that $(\infty+\infty)+(-\infty) = \infty+(-\infty) = 0$ while $\infty+(\infty+(-\infty)) = \infty+0 = \infty \neq 0$. For (VS7ii), note that $(1+(-2))\infty = (-1)\infty = -\infty$ while $1\cdot\infty+(-2)\cdot\infty = \infty+(-\infty) = 0 \neq -\infty$. (The other properties hold.)

Exercise 1.C

We will use the Condition 1.34 of the textbook to check whether the subset is a subspace or not.

1 (c) No. $(1,1,0)$, $(0,1,1)$ are elements of this set since $1\cdot1\cdot0 = 0\cdot1\cdot1 = 0$. Their sum is $(1,2,1)$ which is not in this set since $1\cdot2\cdot1 = 2 \neq 0$. Thus it is not closed under addition.

(d) Yes.

(i) The zero element $(0,0,0)$ is contained in this set: $0 = 5\times0$.

(ii) Let (x_1,x_2,x_3), (y_1,y_2,y_3) be in this set. Then $x_1+y_1 = 5x_3+5y_3 = 5(x_3+y_3)$. Thus their sum $(x_1+y_1, x_2+y_2, x_3+y_3)$ is also in this set.

(iii) For any $a\in\mathbb{F}$, (x_1,x_2,x_3) in this set, $ax_1 = a(5x_3) = 5(ax_3)$. Thus, $a(x_1,x_2,x_3)$ is also in this set.

3 (i) Let f_0 be the zero function, which is the zero element of the space of all real-valued functions on $(-4,4)$. It is differentiable and $f_0'(-1) = 0 = 3f_0(2)$.

(ii) Let f, g be two elements of this subset. By elementary calculus, we know that $f+g$ is differentiable and $(f+g)'(-1) = f'(-1)+g'(-1) = 3f(2)+3g(2) = 3(f+g)(2)$.

(iii) For any $a\in\mathbb{F}$ and any f in this subset, af is differentiable (by calculus again). $(af)'(-1) = af'(-1) = a3f(2) = 3(af)(2)$.

5 No. $(1,0)$ is an element of $\mathbb{R}^2$ and i is an element of $\mathbb{C}$. $i\cdot(1,0) = (i,0)$ which is not in $\mathbb{R}^2$. Therefore it is not closed under scalar multiplication.

7 Consider $U = \{(n,n) : n \text{ is an integer.}\}$. It is a nonempty subset of $\mathbb{R}^2$ as $\overrightarrow{0} = (0,0)\in U$. For any $(m,m),(n,n)\in U$, $(m,m)+(n,n) = (m+n,m+n)\in U$. For any $(n,n)\in U$, $(-n,-n)\in U$ and $(n,n)+(-n,-n) = (0,0)$. So U is closed under addition and under taking additive inverses. However, for $\pi\in\mathbb{R}$, $(1,1)\in U$, $\pi\cdot(1,1) = (\pi,\pi)\notin U$. Therefore the third condition of proposition is violated (it is not closed under scalar multiplication).

9 No. Consider the following functions

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is an integer;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} 1, & \text{if } \pi x \text{ is an integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Note that a real number x is an integer if and only if $x+1$ is an integer. Similarly πx is an integer if and only if $\pi x + 1 = \pi(x + \frac{1}{\pi})$ is an integer. Therefore $f(x) = f(x+1)$ and $g(x) = g(x + \frac{1}{\pi})$ and they belongs to the set of periodic functions. However, the function $h(x) = f(x) + g(x)$ has value 2 when it is evaluated at 0. If $h(x) = 2$ for some $x \in \mathbb{R}$, the value of $f(x)$ and $g(x)$ can only be 1. Thus $x \in \mathbb{Z}$ and $\pi x \in \mathbb{Z}$. If $x \neq 0$, then $\pi = \frac{\pi x}{x} \in \mathbb{Q}$ which is impossible. Therefore for any positive number p, $h(0) \neq h(p)$ and it is not periodic.

12 Let U, W be two subspaces of V.

($\Rightarrow$) If $X := U \cup W$ is a subspace of V, without loss of generality (WLOG), assume $U \not\subset W$. Pick $u \in U \setminus W$. For any $w \in W$, $u + w \in X$. Therefore $u + w \in W$ or U. If it is the first case, $u = (u + w) + (-w) \in W$ since W is a subspace, contradiction. Therefore it is the second case, which implies $w = (u + w) + (-u) \in U$ and $W \subset U$.

($\Leftarrow$) WLOG, assume $U \subset W$. Then $U \cup W = W$ which is assumed to be a subspace of V.

14 Let $v = u + w \in U + W$ where $u = (x_1, x_1, y_1, y_1) \in U$, $w = (x_2, x_2, x_2, y_2) \in W$. Then $v = (x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2) = (x, x, y, z)$ where $x = x_1 + x_2$, $y = y_1 + x_2$, $z = y_1 + y_2 \in \mathbb{F}$.
Now take $(x, x, y, z) \in \mathbb{F}^4$ where $x, y, z \in \mathbb{F}$. Take $x_1 = x$, $y_1 = y$, $x_2 = 0$, $y_2 = z - y$. Then for $u = (x_1, x_1, y_1, y_1) \in U$, $w = (x_2, x_2, x_2, y_2) \in W$, we have $u + w = (x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2) = (x, x, y, z) \in U + W$.

21 Let $W = \{(0, 0, u, v, w) \in \mathbb{F}^5 : u, v, w \in \mathbb{F}\}$. It is a subspace of $\mathbb{F}^5$. Note that $(0, 0, 0, 0, 0) \in W$.
$(0, 0, u_1, v_1, w_1) + (0, 0, u_2, v_2, w_2) = (0, 0, u_1 + u_2, v_1 + v_2, w_1 + w_2) \in W$
for all $(0, 0, u_1, v_1, w_1), (0, 0, u_2, v_2, w_2) \in W$.
$c(0, 0, u, v, w) = (0, 0, cu, cv, cw) \in W$ for all $c \in \mathbb{F}$, $(0, 0, u, v, w) \in W$.
Any element (x, y, u, v, w) in $U \cap W$ must has x and y equal zero and $u = x + y$, $v = x - y$, $w = 2x$, thus u, v, w are zero. This implies that $U \cap W = \{\overrightarrow{0}\}$. Any element (x', y', u', v', w') in $\mathbb{F}^5$ can be written into the sum $(x', y', x'+y', x'-y', 2x') + (0, 0, u'-x'-y', v'-x'+y', w'-2x')$ where the first item is in U and the second one is in W.

23 There are many counterexamples. We only give one below.
Let $V = \mathbb{R}^2$ and let $W = \{(0, y) : y \in \mathbb{R}\}$. Now take $U_1 = \{(x, x) : x \in \mathbb{R}\}$ and take $U_2 = \{(z, 0) : z \in \mathbb{R}\}$. It can be checked that they are all subspaces of V similar to the solution above.
If $v \in U_1 \cap W$, $v = (x, x) = (0, y)$ for some $x, y \in \mathbb{R}$. This implies $x = 0$ and $v = (0, 0)$.
If $v \in U_2 \cap W$, $v = (z, 0) = (0, y)$ for some $z, y \in \mathbb{R}$. This implies $z = 0$, $y = 0$ and $v = (0, 0)$.
Therefore $U_1 \cap W = U_2 \cap W = \{\overrightarrow{0}\}$. For any $v = (a, b) \in V$, $a, b \in \mathbb{R}$, $v = (a, a) + (0, b - a) \in U_1 + W$ and $v = (a, 0) + (0, b) \in U_2 + W$. Hence $V = U_1 \oplus W = U_2 \oplus W$. However, $(1, 0) \in U_2$ but not in U_1. Thus $U_1 \neq U_2$.

24 In the question, U_e and U_o are only given as sets. So we need to check that U_e and U_o are subspaces of $\mathbb{R}^{\mathbb{R}}$ first.

Let $q \in U_e \cap U_o$. Then $q(x) = q(-x) = -q(x)$ for all $x \in \mathbb{R}$. Thus $2q(x) = 0$, $q(x) = 0$ for all $x \in \mathbb{R}$, and hence $U_e \cap U_o = \{\overrightarrow{0}\}$.

Let $f, g \in U_e,\ c \in \mathbb{R}$. Then

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

$$(cf)(-x) = c \cdot f(-x) = c \cdot f(x) = (cf)(x).$$

Let $f, g \in U_o,\ c \in \mathbb{R}$. Then

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x),$$

$$(cf)(-x) = c \cdot f(-x) = c \cdot (-f(x)) = -(cf)(x).$$

Now let $f \in \mathbb{R}^{\mathbb{R}}$. Define function $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in \mathbb{R}$. Then $f_e(x) + f_o(x) = f(x)$. Note that $f_e(-x) = \frac{1}{2}(f(-x) + f(-(-x))) = \frac{1}{2}(f(x) + f(-x)) = f_e(x)$ and $f_o(-x) = \frac{1}{2}(f(-x) - f(-(-x))) = \frac{1}{2}(-f(x) + f(-x)) = -f_o(x)$ for all $x \in \mathbb{R}$. Therefore $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.