THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH1010 University Mathematics 2020-2021 Term 1

Suggested Solutions of Midterm

1. Without using L'Hôpital's rule, evaluate the following limits.

   (a) $\lim_{x\to 0} \frac{\ln(1+3x^2)}{x\sin 2x}$

   (b) $\lim_{x\to\infty} (x - \sqrt{x^2-6x+1})$

   (c) $\lim_{x\to\infty} \frac{e^{\cos x}+e^{\sin x}}{\ln(1+x)}$

   Solution:

   (a)

$$\begin{aligned}\lim_{x\to 0} \frac{\ln(1+3x^2)}{x\sin 2x} &= \lim_{x\to 0} \frac{\ln(1+3x^2)}{3x^2}\cdot\frac{x}{x}\cdot\frac{2x}{\sin 2x}\cdot\frac{3x^2}{2x^2} \\ &= \lim_{x\to 0} \frac{\ln(1+3x^2)}{3x^2}\cdot\frac{2x}{\sin 2x}\cdot\frac{3}{2} \\ &= 1\cdot 1\cdot\frac{3}{2} \\ &= \frac{3}{2}.\end{aligned}$$

   (b)

$$\begin{aligned}\lim_{x\to+\infty} (x - \sqrt{x^2-6x+1}) &= \lim_{x\to+\infty} \frac{x^2-(x^2-6x+1)}{x+\sqrt{x^2-6x+1}} \\ &= \lim_{x\to+\infty} \frac{6x-1}{x+\sqrt{x^2-6x+1}} \\ &= \lim_{x\to+\infty} \frac{6-\frac{1}{x}}{1+\sqrt{1-\frac{6}{x}+\frac{1}{x^2}}} \qquad (\text{since } \sqrt{x^2}=x \text{ if } x>0) \\ &= \frac{6}{1+1} \\ &= 3.\end{aligned}$$

   (c) Note that, for any $x\in\mathbb{R}$,

$$-1\le\cos x\le 1 \quad \text{and} \quad -1\le\sin x\le 1.$$

Since e^x is an increasing function, we have, for any $x \in \mathbb{R}$,

$$e^{-1} \le e^{\cos x} \le e \quad \text{and} \quad e^{-1} \le e^{\sin x} \le e.$$

Hence,

$$\frac{2e^{-1}}{\ln(1+x)} \le \frac{e^{\cos x} + e^{\sin x}}{\ln(1+x)} \le \frac{2e}{\ln(1+x)} \quad \text{for } x > 0.$$

Since

$$\lim_{x\to+\infty} \frac{2e^{-1}}{\ln(1+x)} = \lim_{x\to+\infty} \frac{2e}{\ln(1+x)} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to+\infty} \frac{e^{\cos x} + e^{\sin x}}{\ln(1+x)} = 0.$$

2. Find $\dfrac{dy}{dx}$ if

(a) $y = (\tan x)\ln(1+\sin 2x)$

(b) $y = \tan^{-1}(e^x)$

(c) $x\sin y + y^2\cos x = 1$

(d) $y = (1+x)^{\sqrt{x}}$

Solution:

(a) $\dfrac{dy}{dx} = (\sec^2 x)\ln(1+\sin 2x) + (\tan x)\cdot\dfrac{2\cos 2x}{1+\sin 2x}$.

(b) $\dfrac{dy}{dx} = \dfrac{1}{1+(e^x)^2}\cdot e^x = \dfrac{e^x}{1+e^{2x}}$.

(c) Differentiate both sides with respect to x,

$$\begin{aligned}\frac{d}{dx}(x\sin y + y^2\cos x) &= \frac{d}{dx}(1)\\ \sin y + x\cos y\frac{dy}{dx} + 2y\frac{dy}{dx}\cos x + y^2(-\sin x) &= 0\\ \frac{dy}{dx} &= \frac{y^2\sin x - \sin y}{2y\cos x + x\cos y}.\end{aligned}$$

(d) Take logarithm and then differentiate both sides with respect to x,

$$\begin{aligned}\frac{d}{dx}\ln y &= \frac{d}{dx}(\sqrt{x}\ln(1+x))\\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{2\sqrt{x}}\ln(1+x) + \frac{\sqrt{x}}{1+x}\\ \frac{dy}{dx} &= (1+x)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln(1+x) + \frac{\sqrt{x}}{1+x}\right).\end{aligned}$$

3. Let

$$f(x) = \begin{cases} x^2 \cos(\ln|x|), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$.

(b) Find $f'(0)$.

(c) Determine whether $f'(x)$ is continuous at $x = 0$.

Solution:

(a) Note that

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \quad \text{for } x \neq 0.$$

Hence, for $x \neq 0$,

$$f'(x) = 2x\cos(\ln|x|) + x^2(-\sin(\ln|x|)) \cdot \frac{1}{x} = 2x\cos(\ln|x|) - x\sin(\ln|x|).$$

(b)

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos(\ln|x|) - 0}{x} = \lim_{x \to 0} x\cos(\ln|x|) = 0,$$

since $|\cos(\ln|x|)| \leq 1$ for $x \neq 0$ and $\lim_{x \to 0} x = 0$. Hence $f'(0) = 0$.

(c) Since $|\cos(\ln|x|)| \leq 1$, $|\sin(\ln|x|)| \leq 1$ for $x \neq 0$ and $\lim_{x \to 0} x = 0$, we have

$$\lim_{x \to 0} x\cos(\ln|x|) = 0 \quad \text{and} \quad \lim_{x \to 0} x\sin(\ln|x|) = 0.$$

Thus

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} [2x\cos(\ln|x|) - x\sin(\ln|x|)] = 0 = f'(0).$$

Hence $f'(x)$ is continuous at $x = 0$.

4. Let a_n be the sequence defined by

$$\begin{cases} a_{n+1} = 5 - \dfrac{1}{a_n}, \text{ for } n \geq 1 \\ a_1 = 1. \end{cases}$$

(a) Show that $a_n \leq 5$ for any $n \geq 1$.

(b) Show that a_n is convergent.

(c) Find $\lim_{n\to\infty} a_n$.

Solution:

(a) Let $P(n)$ be the statement "$1 \leq a_n \leq 5$".

- When $n = 1$, $1 \leq a_1 = 1 \leq 5$. Therefore $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n \geq 1$, i.e. $1 \leq a_n \leq 5$. Then,

$$a_{n+1} = 5 - \frac{1}{a_n} \leq 5 - \frac{1}{5} = \frac{24}{5} \leq 5,$$

and

$$a_{n+1} = 5 - \frac{1}{a_n} \geq 5 - \frac{1}{1} = 4 \geq 1.$$

Therefore, $P(n+1)$ is true.

By mathematical induction, $1 \leq a_n \leq 5$ for all natural numbers n.

(If we only consider "$a_n \leq 5$" in induction, then a_n could be negative, and it is possible that

$$a_{n+1} = 5 - \frac{1}{a_n} > 5.$$

On the other hand, to show that $a_{n+1} = 5 - \frac{1}{a_n} > 0$, we need $a_n > \frac{1}{5}$. The lower bound 1 here is chosen for convenience.)

(b) Let $Q(n)$ be the statement "$a_{n+1} \geq a_n$".

- When $n = 1$, $a_2 = 5 - \frac{1}{1} = 4 \geq 1 = a_1$. Therefore $Q(1)$ is true.
- Suppose $Q(n)$ is true for some natural number $n \geq 1$, i.e. $a_{n+1} \geq a_n$. Then, since $a_n > 0$, we have

$$a_{n+2} = 5 - \frac{1}{a_{n+1}} \geq 5 - \frac{1}{a_n} = a_{n+1}.$$

Therefore, $Q(n+1)$ is true.

By mathematical induction, $a_{n+1} \geq a_n$ for all natural numbers n.

Since $\{a_n\}$ is monotonic increasing and bounded above, it follows from the Monotone Convergence theorem that $\{a_n\}$ is convergent.

(Note that $a_{n+1} \geq a_n > 0$ is need to conclude that

$$5 - \frac{1}{a_{n+1}} \geq 5 - \frac{1}{a_n}.$$

For example, $2 > -1$ but $-\frac{1}{2} < -\frac{1}{-1}$.)

(c) Let $\ell = \lim_{n\to\infty} a_n$. By (a), we have $1 \leq \ell \leq 5$. Letting $n \to \infty$, we have

$$\ell = 5 - \frac{1}{\ell}$$
$$\ell^2 - 5\ell + 1 = 0$$
$$\ell = \frac{5+\sqrt{21}}{2} \quad \text{or} \quad \ell = \frac{5-\sqrt{21}}{2} \quad \text{(rejected)}.$$

Hence $\lim_{n\to\infty} a_n = \frac{5+\sqrt{21}}{2}$.

End of paper