Tutorial 8

* (Total) differentiability $\rightsquigarrow$ $\exists$ "good linear approximation"

$\rightsquigarrow$ Chain rule

$\rightsquigarrow$ $\nabla f(\vec{a}) \cdot \vec{u} = D_{\vec{u}} f(\vec{a})$

A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if (def.) $\exists\ k \in \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = k$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - kh}{h} = 0$$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0,$$

where $T: \mathbb{R} \to \mathbb{R}$ is a linear transformation.
$h \mapsto kh$

A natural generalization to higher dimensions:

A function $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a} \in \mathbb{R}^n$ if (def.) $\exists$ linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

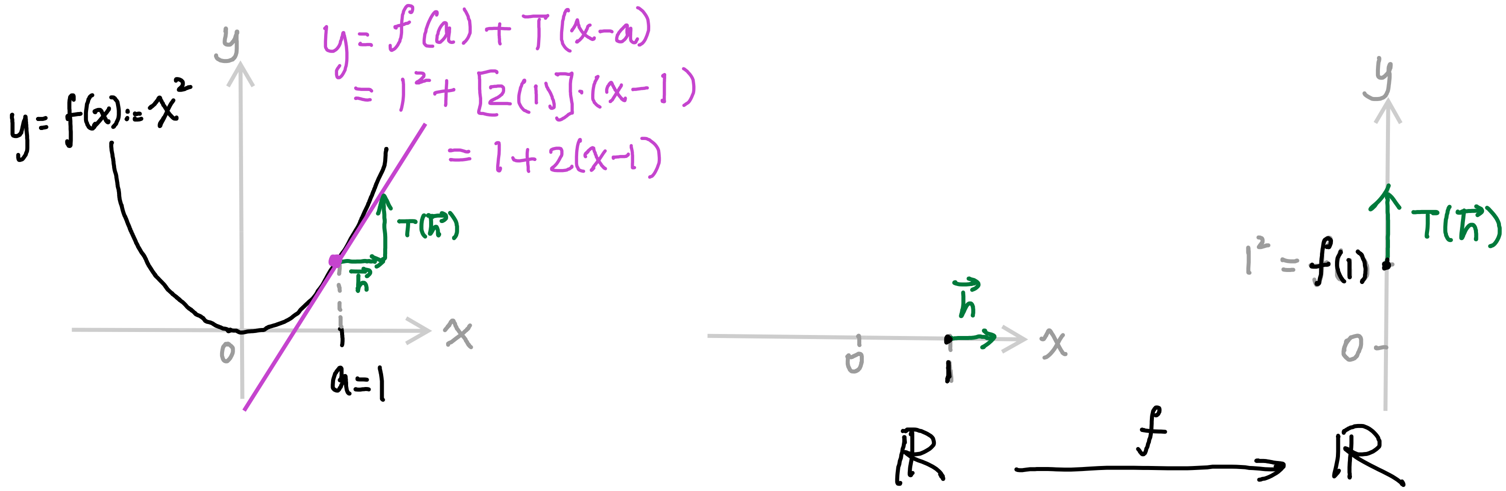
$$\lim_{\vec{h} \to \vec{0}} \frac{\| \vec{f}(\vec{a}+\vec{h}) - \vec{f}(\vec{a}) - T(\vec{h}) \|}{\| \vec{h} \|} = 0$$

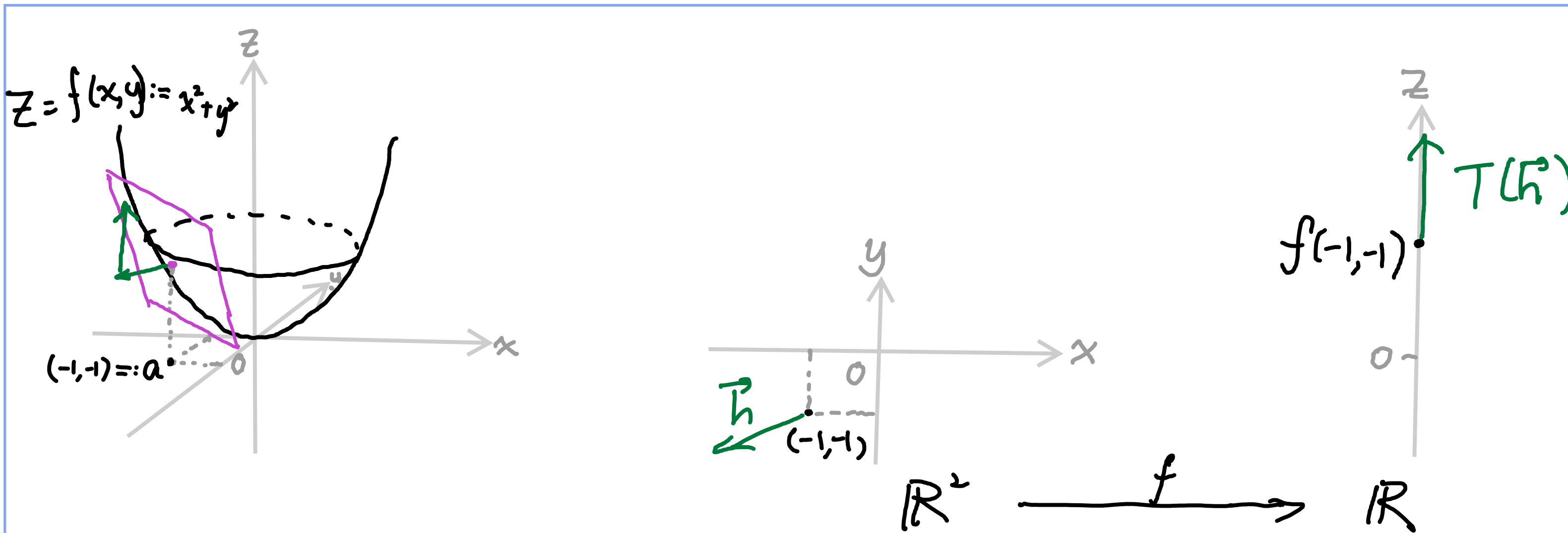
It could be shown that such linear transformation is unique and its matrix representation is the "matrix of partial derivatives", called Jacobian matrix of $\vec{f}$ at $\vec{a}$ in lecture notes.

We may view the numerator in two ways:

$$\vec f(\vec a+\vec h) - \underbrace{\left(\vec f(\vec a) + T(\vec h)\right)}_{\text{linearization of } \vec f \text{ at } \vec a}$$

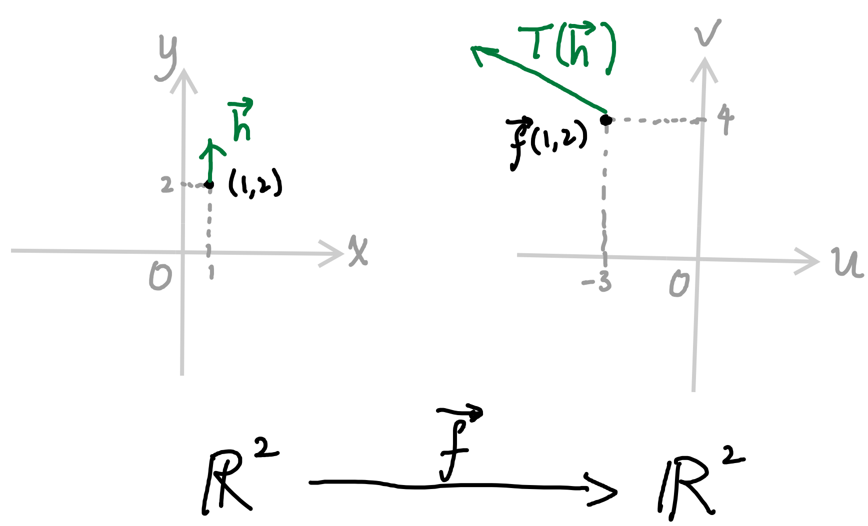
$$= \underbrace{\left(\vec f(\vec a+\vec h) - \vec f(\vec a)\right)}_{\text{change in } \vec f} - T(\vec h)$$





$$\vec f(x,y) := (x^2-y^2,\ 2xy)$$

?



Using the notations in lecture note : Given $f: \mathbb{R}^n \to \mathbb{R}^m$,

$$\underbrace{T}_{\substack{\text{linear}\\ \text{transformation}}}(\vec{h}) = \underbrace{\left(D\vec{f}(\vec{a})\right)}_{\substack{\text{Jacobian}\\ \text{matrix}}}(\vec{h}) = \begin{pmatrix} \text{—} \ \vec{\nabla} f_1(\vec{a}) \ \text{—} \\ \vdots \\ \text{—} \ \vec{\nabla} f_m(\vec{a}) \ \text{—} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

① Let $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined as follows:

$\forall (x,y) \in \mathbb{R}^2, \quad \vec{f}(x,y) = \left(\sqrt{xy}, \sqrt{\frac{y}{x}}\right).$

(a) Find the Jacobian matrix of $\vec{f}$ at $(2,8)$.

(b) Use the linearization of $\vec{f}$ at $(2,8)$ to approximate $\vec{f}(1.9, 8.2)$

Ans: 1 (a) Let $f_1(x,y) := \sqrt{xy}$ and $f_2(x,y) := \sqrt{\frac{y}{x}}$.

$$D\vec{f}(x,y) = \begin{pmatrix} \text{—} \ \vec{\nabla} f_1(x,y) \ \text{—} \\ \text{—} \ \vec{\nabla} f_2(x,y) \ \text{—} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ -\frac{1}{2}\sqrt{y}\left(x^{-\frac{3}{2}}\right) & \frac{1}{2\sqrt{xy}} \end{pmatrix}$$

$$\therefore D\vec{f}(2,8) = \begin{pmatrix} 1 & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{8} \end{pmatrix} = T$$

1(b) $\left(\vec{f}(\vec{a}+\vec{h}) \approx \vec{f}(\vec{a}) + T(\vec{h}) \right)$

$$\begin{aligned} \vec{f}(1.9, 8.2) &= \vec{f}\left((2,8) + (-0.1, +0.2)\right) \\ &\approx \vec{f}(2,8) + \begin{pmatrix} 1 & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} -0.1 \\ +0.2 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -0.1 + 0.05 \\ 0.05 + 0.025 \end{pmatrix} = \begin{pmatrix} 3.95 \\ 2.075 \end{pmatrix} \end{aligned}$$

② A twice continuously differentiable (i.e. C^2) function $f:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ is a harmonic function if it satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

Show the Laplace equation in polar coordinates, i.e.,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}=0$$

Ans:

Let $\varphi:(0,\infty)\times\mathbb{R}\longrightarrow\mathbb{R}^2\setminus\{0\}$

$$(r,\theta)\longmapsto(r\cos\theta, r\sin\theta)$$

By Chain rule (why can we use this?),

$$D(f\circ\varphi)(r_0,\theta_0)=Df(\varphi(r_0,\theta_0))\cdot D\varphi(r_0,\theta_0)$$

$$\left(\frac{\partial(f\circ\varphi)}{\partial r}(r_0,\theta_0),\ \frac{\partial(f\circ\varphi)}{\partial\theta}(r_0,\theta_0)\right)=\left(\frac{\partial f}{\partial x}(\varphi(r_0,\theta_0)),\ \frac{\partial f}{\partial y}(\varphi(r_0,\theta_0))\right)\cdot\begin{pmatrix}\frac{\partial\varphi_1}{\partial r}(r_0,\theta_0) & \frac{\partial\varphi_1}{\partial\theta}(r_0,\theta_0)\\ \frac{\partial\varphi_2}{\partial r}(r_0,\theta_0) & \frac{\partial\varphi_2}{\partial\theta}(r_0,\theta_0)\end{pmatrix}$$

With abuse of notation, we have

$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial\theta}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)=\frac{1}{r}\left(\frac{\partial f}{\partial r}+r\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)\right)$$

$$=\frac{1}{r}\left[\left(\frac{\partial f}{\partial x}\cdot\cos\theta+\frac{\partial f}{\partial y}\cdot\sin\theta\right)+r\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cdot\cos\theta+\frac{\partial f}{\partial y}\sin\theta\right)\right]$$

$$=\frac{1}{r}\Big[\left(\frac{\partial f}{\partial x}\cdot\cos\theta+\frac{\partial f}{\partial y}\cdot\sin\theta\right)$$

$$+r\cos\theta\left(\frac{\partial^2 f}{\partial x^2}\cdot\cos\theta+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\sin\theta\right)+r\sin\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\cos\theta+\frac{\partial^2 f}{\partial y^2}\cdot\sin\theta\right)\Big]$$

$$\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} = \frac{1}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial\theta}\right)$$

$$= \frac{1}{r^2}\cdot\left[\left(\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial x}\cdot\frac{\partial^2 x}{\partial\theta^2} + \left(\frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial y}{\partial\theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial^2 y}{\partial\theta^2}\right)\right)\right]$$

$$= \frac{1}{r^2}\left[\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial\theta}\cdot\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial y}{\partial\theta}\right)\cdot\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial x}\cdot\frac{\partial}{\partial\theta}(-r\sin\theta)\right.$$

$$\left.+\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial\theta}\right)\cdot\frac{\partial y}{\partial\theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial\theta}(r\cos\theta)\right]$$

$$= \frac{1}{r^2}\left[\left(\frac{\partial^2 f}{\partial x^2}\cdot(-r\sin\theta) + \underline{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot(r\cos\theta)}\right)\cdot(-r\sin\theta) + \underline{\frac{\partial f}{\partial x}\cdot(-r\cos\theta)}\right.$$

$$\left.+\left(\underline{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot(-r\sin\theta)} + \frac{\partial^2 f}{\partial y^2}\cdot(r\cos\theta)\right)(r\cos\theta) + \underline{\frac{\partial f}{\partial y}(-r\sin\theta)}\right]$$

$$\therefore \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta\right) + \left(\frac{\partial^2 f}{\partial x^2}\sin^2\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$