Tutorial 6

* Continuous extension
* Partial derivatives
* Differentiability

① Let $f(x,y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$

(a) Find a possible domain of f such that $(0,0)$ is a cluster point of the domain.

(b) Could you extend f such that the new function obtained is continuous at $(0,0)$?

Rivision: Compare and contrast the definitions for limit and continuity:

Let $A \subseteq \mathbb{R}^n$. Let $\vec{a} \in \mathbb{R}^n$ be a cluster point of A.
Let $\vec{f}: A \to \mathbb{R}^m$ be a function.
We say that $\boxed{\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) \text{ exists and equals } \vec{L}}$ iff
$\forall \varepsilon > 0, \exists \delta > 0$ such that
if $\vec{x} \in B_\delta(\vec{a}) \cap A \setminus \{\vec{a}\}$, then $\vec{f}(\vec{x}) \in B_\varepsilon(\vec{L})$.

Let $A \subseteq \mathbb{R}^n$. Let $\vec{a} \in A$.
Let $f: A \to \mathbb{R}$ be a function.
We say that $\boxed{f \text{ is continuous at } \vec{a}}$ iff
$\forall \varepsilon > 0, \exists \delta > 0$ such that
if $\vec{x} \in B_\delta(\vec{a}) \cap A$, then $f(\vec{x}) \in B_\varepsilon(f(\vec{a}))$

Remark: If $\vec{a}$ is a cluster point of $A \subseteq \mathbb{R}^n$ and $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$, then f is continuous at $\vec{a}$.

Ans: 1(a) Let $h(x,y) := \dfrac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}$.

We have $f(x,y) = \ln(h(x,y))$.

"largest domain of f" $= \{(x,y) \in \mathbb{R}^2 : h(x,y) > 0 \text{ and } x^2+y^2 \neq 0\}$

Let $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. $\exists\, r \in [0,\infty)$ and $\theta \in \mathbb{R}$ such that $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
&(h(x,y) > 0 \text{ and } x^2+y^2 \neq 0) \\
\Longleftrightarrow\ &(h(r\cos\theta, r\sin\theta) > 0 \text{ and } r^2 \neq 0) \\
\Longleftrightarrow\ &\left(\frac{3r^2 - r^4\cos^2\theta\sin^2\theta}{r^2} > 0 \text{ and } r > 0\right) \\
\Longleftrightarrow\ &(3 - r^2\cos^2\theta\sin^2\theta > 0 \text{ and } r > 0) \\
\Longleftarrow\ &\quad r \in (0, \sqrt{3})
\end{aligned}$$

$D := B_{\sqrt{3}}((0,0)) \setminus \{(0,0)\} = \{(x,y) \in \mathbb{R}^2 : 0 < \sqrt{x^2+y^2} < \sqrt{3}\}$

is a possible domain of f. It is not hard to check that $(0,0)$ is a cluster point of D.

1(b) Yes. By Tutorial 5 ②, we have

$\lim_{(x,y)\to(0,0)} h(x,y)$ exists and equals 3.

Define $\tilde{h}(x,y) := \begin{cases} 3 & \text{if } (x,y) = (0,0) \\ h(x,y) & \text{otherwise} \end{cases}$

$$\tilde{f}(x,y) := \begin{cases} \ln(3) & \text{if } (x,y) = (0,0) \\ f(x,y) & \text{otherwise} \end{cases} = \ln(\tilde{h}(x,y))$$

By Thm on P.6 Week 5 notes,

$$\lim_{(x,y)\to(0,0)} \tilde{f}(x,y) = \ln\left(\lim_{(x,y)\to(0,0)} \tilde{h}(x,y)\right) = \ln(3) = \tilde{f}(0,0)$$

$\therefore \tilde{f}$ is continuous at $(0,0)$.

2 (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function.

(i) If $\frac{\partial f}{\partial y}(x,y) \equiv 0$, show that f is independent of the second variable.

(ii) If $\frac{\partial f}{\partial x}(x,y) \equiv 0 \equiv \frac{\partial f}{\partial y}(x,y)$, show that f is constant.

(b) Let $S := \{ (x,y) \in \mathbb{R}^2 : x < 0 \text{ or } (x \geq 0 \text{ and } y \neq 0) \}$.

(i) Let $f : S \to \mathbb{R}$ be a function.

If $\frac{\partial f}{\partial x}(x,y) \equiv 0 \equiv \frac{\partial f}{\partial y}(x,y)$, show that f is constant.

(ii) Find a function $f : S \to \mathbb{R}$ such that $\frac{\partial f}{\partial y}(x,y) \equiv 0$ but f is not independent of the second variable.

Ans: 2(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function.

(i) Suppose $\frac{\partial f}{\partial y}(x,y) \equiv 0$.

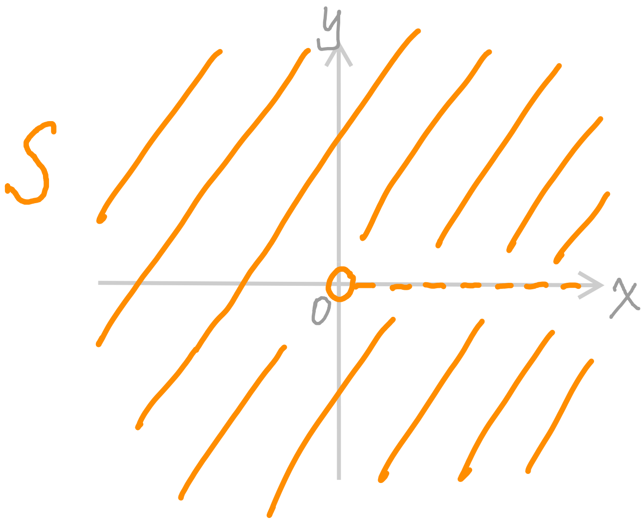
Let $x \in \mathbb{R}$. Let $y_1, y_2 \in \mathbb{R}$. WLOG, assume $y_1 < y_2$.
By Mean Value Theorem, there exists $c_x \in (y_1, y_2)$ such that $f(x, y_2) - f(x, y_1) = \frac{\partial f}{\partial y}(x, c_x)(y_2 - y_1)$.
Since $\frac{\partial f}{\partial y}(x, c_x) = 0$, $f(x, y_1) = f(x, y_2)$.
Hence, f is independent of y.

(ii) Suppose $\frac{\partial f}{\partial x}(x,y) \equiv 0 \equiv \frac{\partial f}{\partial y}(x,y)$. By (i), f is independent of both x and y. Thus, for each $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$f(x_1, y_1) = f(x_1, y_2) = f(x_2, y_2).$$

Hence, f is constant.

Ans: 2(b) Let $S = \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ or } (x \geq 0 \text{ and } y \neq 0)\}$.



(i) Suppose $f : S \to \mathbb{R}$ is a function such that $\frac{\partial f}{\partial x}(x,y) \equiv 0 \equiv \frac{\partial f}{\partial y}(x,y)$.

Since each pair of points in S can be connected by a finite sequence of line segments each parallel to one of the axes, we could use Mean Value Theorem to show that f is constant.

(ii) Define $f : S \to \mathbb{R}$ as follows:

$$f(x,y) := \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ x & \text{otherwise} \end{cases}$$

It is not hard to check that $\frac{\partial f}{\partial y}(x,y) \equiv 0$.

$$\left(\begin{array}{l} f(10,1) = 10 \\ f(10,-1) = 0 \neq f(10,1) \end{array} \right) \Rightarrow f \text{ is not independent of the second variable.}$$

③ Let $f(x,y) = x + y$.

(a) Show that $\displaystyle\lim_{(x,y) \to (1,2)} \frac{f(x,y) - f(1,2)}{\sqrt{(x-1)^2 + (y-2)^2}}$ does not exist.

(b) Could we conclude (from (a)) that f is not differentiable at $(x,y) = (1,2)$?

(c) Show by definition that f is differentiable at $(x,y) = (1,2)$.

Ans: 3(a)

$$\lim_{\substack{(x,y)\to(1,2) \\ \text{along } x=1}} \frac{f(x,y)-f(1,2)}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{y\to 2} \frac{(1+y)-(1+2)}{\sqrt{(y-2)^2}}$$

$$= \lim_{y\to 2} \frac{y-2}{|y-2|}$$

$$\lim_{y\to 2^+} \frac{y-2}{|y-2|} = \lim_{y\to 2^+} \frac{y-2}{y-2} = \lim_{y\to 2^+} 1 = 1$$

$$\lim_{y\to 2^-} \frac{y-2}{|y-2|} = \lim_{y\to 2^-} \frac{y-2}{-(y-2)} = \lim_{y\to 2^-} (-1) = -1 \neq 1$$

$\therefore$ The limit does not exist.

(b) No.

(c) $\vec{\nabla} f(1,2) = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)\right) = (1, 1).$

$$\lim_{(x,y)\to(1,2)} \frac{\varepsilon(x,y)}{\|(x,y)-(1,2)\|} = \lim_{(x,y)\to(1,2)} \frac{f(x,y)-f(1,2)-(1,1)\cdot((x,y)-(1,2))}{\|(x,y)-(1,2)\|}$$

$$= \lim_{(x,y)\to(1,2)} \frac{(x+y)-(1+2)-(1\cdot(x-1)+1\cdot(y-2))}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(x,y)\to(1,2)} \frac{0}{\sqrt{(x-1)^2+(y-2)^2}}$$

$$= \lim_{(x,y)\to(1,2)} 0 = 0$$

$\therefore$ f is differentiable at $(x,y)=(1,2)$.