Tutorial 2

This tutorial focuses mainly on finding distances between point(s) and linear object(s) in $\mathbb{R}^3$.

* Familiarise yourself with the "conversion"

  Equation form $\longleftrightarrow$ Parametric form

* "Check" that the two linear objects do not intersect with each other.

The following table summarises some ways of finding distances between point(s), line(s) and plane(s).

	$\vec{q}$	$\vec{d}$, $\vec{r}(t)$	$\vec{n}$, $\pi(x,y,z)=0$
$\vec{p}$	**1** $\lVert \vec{p}-\vec{q} \rVert$	**2** $(\vec{p}-\vec{r}(t))\cdot\vec{d}=0$ Solve for t and Compute $\lVert \vec{p}-\vec{r}(t) \rVert$	**3** $\pi(\vec{p}+t\vec{n})=0$ Solve for t and Compute $\lVert t\vec{n} \rVert$
ℓ		**4** Case 1: parallel lines Pick a point on ℓ and see $\boxed{2}$. Case 2: skew lines. Set up two equations with two unknowns	**5** Pick a point on ℓ and see $\boxed{3}$.
P			**6** Pick a point on P and see $\boxed{3}$

Let's try to work on some examples.

1. Given two straight lines in $\mathbb{R}^3$:

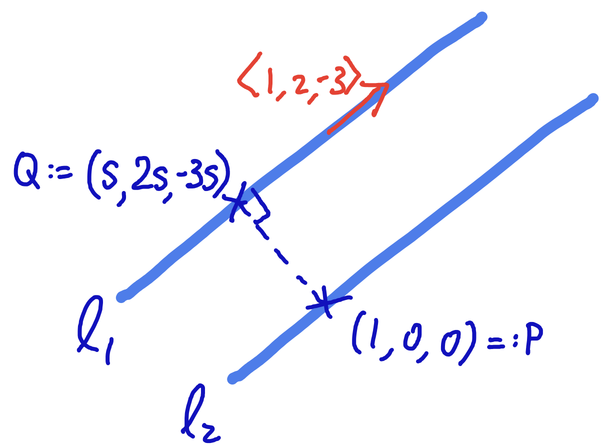
$$\ell_1 : (x,y,z) = s(1,2,-3) \quad \forall s \in \mathbb{R};$$

and $$\ell_2 : (x,y,z) = (1,0,0) + t(-1,-2,3) \quad \forall t \in \mathbb{R}.$$

Show that ℓ_1 and ℓ_2 are parallel and find the distance between them.

Ans: Since $(1,2,-3) = -(-1,-2,3)$, the direction vectors of ℓ_1 and ℓ_2 are parallel.

Thus, ℓ_1 and ℓ_2 are parallel.



To find the distance between ℓ_1 and ℓ_2, pick a point on a line first, say $(1,0,0)$ on ℓ_2, and express the closest point on ℓ_1 in terms of the parameter.

Let $P := (1,0,0)$ and Q be a point on ℓ_1 that is closest to P. Since Q lies in ℓ_1, there exists $s \in \mathbb{R}$ such that $Q = (s, 2s, -3s)$.

As $\overrightarrow{PQ} \perp \langle 1,2,-3 \rangle$, $\overrightarrow{PQ} \cdot \langle 1,2,-3 \rangle = 0$

$$\begin{aligned} \langle s-1, 2s-0, -3s-0 \rangle \cdot \langle 1,2,-3 \rangle &= 0 \\ (s-1) + 4s + 9s &= 0 \\ s &= \frac{1}{14} \end{aligned}$$

$\therefore$ Distance between ℓ_1 and ℓ_2 $= \| \overrightarrow{PQ} \|$

$$= \sqrt{\left(\frac{1}{14}-1\right)^2 + \left(\frac{2}{14}\right)^2 + \left(\frac{-3}{14}\right)^2} = \sqrt{\frac{13}{14}}$$

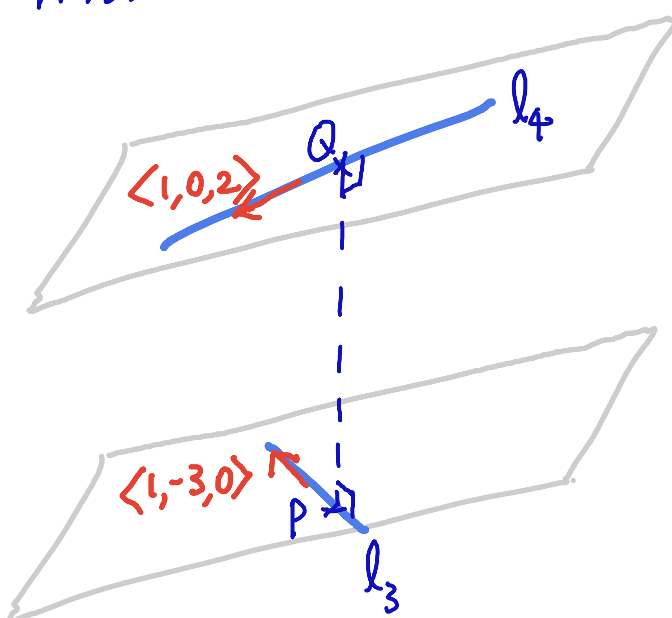
② Given two skew lines in $\mathbb{R}^3$
(i.e. non-parallel lines that do not intersect one another):

$$\ell_3 : (x,y,z) = (1,0,3) + s(1,-3,0) \quad \forall s \in \mathbb{R};$$

and $$\ell_4 : (x,y,z) = (0,0,2) + t(1,0,2) \quad \forall t \in \mathbb{R}.$$

Find the distance between ℓ_3 and ℓ_4.

Ans:



Let P be a point in ℓ_3 and Q be a point in ℓ_4 such that $\|\overrightarrow{PQ}\|$ is the distance between ℓ_3 and ℓ_4.

There exist $s, t \in \mathbb{R}$ such that

$P = (1+s, -3s, 3)$ and

$Q = (t, 0, 2+2t)$.

To solve for the two unknowns, we may want to set up two equations, which come from the "observation" that $\overrightarrow{PQ} \perp \ell_3$ and $\overrightarrow{PQ} \perp \ell_4$

$$\overrightarrow{PQ} = (t-s-1, 3s, 2t-1)$$

$$\begin{cases} \overrightarrow{PQ} \cdot \langle 1,0,2 \rangle = 0 \\ \overrightarrow{PQ} \cdot \langle 1,-3,0 \rangle = 0 \end{cases}$$

$$\begin{cases} (t-s-1) + (4t-2) = 0 \quad \text{——— } (1) \\ (t-s-1) - 9s = 0 \quad \text{——— } (2) \end{cases}$$

From (1), $s = 5t-3$ ——— (3)

Sub. (3) into (2), $2 - 4t - 45t + 27 = 0 \Rightarrow t = \frac{29}{49}$, $s = \frac{-2}{49}$

$$\therefore \|\overrightarrow{PQ}\| = \left\| \left(\frac{-18}{49}, \frac{-6}{49}, \frac{9}{49} \right) \right\| = \frac{1}{49}\sqrt{324+36+81} = \frac{3}{7}$$ //

③ (a) Find the equation of the plane Π containing the straight line

$$L: \frac{x-4}{2} = \frac{y-3}{5} = \frac{z+1}{-2}$$

and the point $P(2, -4, 2)$

Ans: Let $t = \frac{x-4}{2} = \frac{y-3}{5} = \frac{z+1}{-2}$. We have

$$\begin{cases} x = 4+2t \\ y = 3+5t \\ z = -1-2t \end{cases} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} t.$$

$\therefore$ The point $Q := (4, 3, -1)$ and the direction vector $\vec{v} := (2, 5, -2)$ lie in the line L, and thus the plane Π.

A normal vector of Π is

$$\vec{n} = \overrightarrow{PQ} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 7 & -3 \\ 2 & 5 & -2 \end{vmatrix}$$

$$= \hat{i} - 2\hat{j} - 4\hat{k}$$

Sub. $(x, y, z) = (2, -4, 2)$ into $x - 2y - 4z + D = 0$,

$$2 - 2(-4) - 4(2) + D = 0$$

$$D = -2$$

$\therefore$ Equation of Π is $x - 2y - 4z - 2 = 0$.

③ (b) Find the distance between Π and

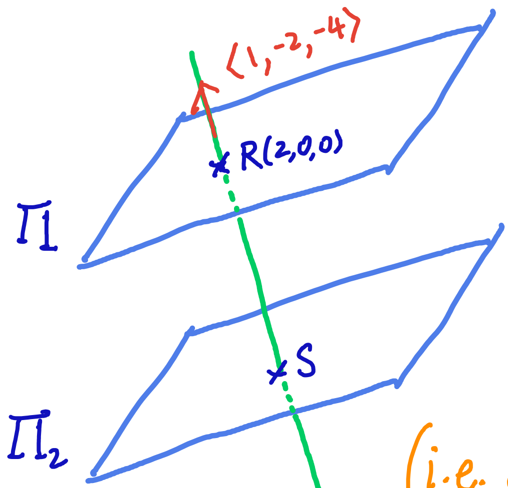
$$\Pi_2: -2x+4y+8z+1=0.$$

Ans: Note that $\Pi: x-2y-4z-2=0$ and

$$\Pi_2: -2x+4y+8z+1=0$$

are parallel planes because

$$-2(1,-2,-4)=(-2,4,8).$$



Pick a point in Π, say $R:=(2,0,0)$.

pick some "convenient" point that satisfies eqn for Π

Let $S:=(2,0,0)+t(1,-2,-4)$ be a point that lies in Π_2.

(i.e. S is the (orthogonal) projection of R on Π_2)

$$-2(2+t)+4(-2t)+8(-4t)+1=0$$

$$-42t-3=0$$

$$t=\frac{1}{-14}$$

$\therefore$ Distance between Π and Π_2 is

$$\|\overrightarrow{RS}\| = \left\|\frac{1}{-14}(1,-2,-4)\right\| = \frac{1}{14}\sqrt{21}$$

Remark: The above example uses the method stated on the first page. There is another (possibly easier) method which pick arbitrary points $A\in\Pi$ and $B\in\Pi_2$, and then compute $|\text{proj}_{\vec{n}}\overrightarrow{AB}| = \left|\overrightarrow{AB}\cdot\frac{\vec{n}}{\|\vec{n}\|}\right|$, where $\vec{n}$ is a normal vector of Π or Π_2.