THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Suggested Solution of Coursework 9

(1) Compute

$$\int e^{\sin x}\cos x\,dx$$
$$\int \frac{1}{x\ln x}\,dx$$

Solutions:

$$\begin{aligned}&\int e^{\sin x}\cos x\,dx\\ =&\int e^{\sin x}\,d\sin x\\ =&\,e^{\sin x}+C\end{aligned}$$

$$\begin{aligned}&\int \frac{1}{x\ln x}\,dx\\ =&\int \frac{1}{\ln x}\,d\ln x\\ =&\ln|\ln x|+C\end{aligned}$$

(2) Evaluate the integral using an appropriate substitution

$$\int \sin(2\sin(t))\cos(t)\,dt$$

Solutions:
Sub $u=\sin(t), du=\cos(t)dt$

$$\begin{aligned}&\int \sin(2\sin(t))\cos(t)\,dt\\ =&\int \sin(2u)\,du\\ =&\frac{-\cos(2u)}{2}+C\\ =&\frac{-\cos(2\sin(t))}{2}+C\end{aligned}$$

(3) Evaluate the integral

$$\int \frac{\cos(8x)}{1+\sin^2(8x)}\,dx$$

Solutions:

$$\begin{aligned}
&\int \frac{\cos(8x)}{1+\sin^2(8x)}\,dx \\
&= \int \frac{1}{8}\cdot\frac{1}{1+\sin^2(8x)}\,d\sin(8x) \\
&= \frac{1}{8}\arctan(\sin(8x)) + C
\end{aligned}$$

(4) Evaluate the indefinite integral

$$\int x^5 \sin(x^6)\,dx$$

Solutions:

$$\begin{aligned}
&\int x^5 \sin(x^6)\,dx \\
&= \int \frac{\sin(x^6)}{6}\,dx^6 \\
&= -\frac{\cos(x^6)}{6} + C
\end{aligned}$$

(5) The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2+1)e^{x^2}$ does. Evaluate

$$\int (2x^2+1)e^{x^2}\,dx$$

Solutions:

$$\begin{aligned}
&\int (2x^2+1)e^{x^2}\,dx \\
&= \int 2x^2 e^{x^2}\,dx + \int e^{x^2}\,dx \\
&= \int x\,de^{x^2} + \int e^{x^2}\,dx \\
&= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \\
&= xe^{x^2} + C
\end{aligned}$$

(6) Evaluate the indefinite integral

$$\int e^{6x}\sin(7x)\,dx$$

Solutions:

$$\begin{aligned}
&\int e^{6x}\sin(7x)\,dx \\
&= \int -\frac{1}{7}e^{6x}\,d\cos(7x) \\
&= -\frac{1}{7}e^{6x}\cos(7x) + \int \frac{1}{7}\cos(7x)\,de^{6x} \\
&= -\frac{1}{7}e^{6x}\cos(7x) + \int \frac{6}{49}e^{6x}\,d\sin(7x) \\
&= -\frac{1}{7}e^{6x}\cos(7x) + \frac{6}{49}e^{6x}\sin(7x) - \int \frac{6}{49}\sin(7x)\,de^{6x} \\
&= -\frac{1}{7}e^{6x}\cos(7x) + \frac{6}{49}e^{6x}\sin(7x) - \int \frac{36}{49}e^{6x}\sin(7x)\,dx
\end{aligned}$$

Then

$$\begin{aligned}
\int e^{6x}\sin(7x)\,dx + \int \frac{36}{49}e^{6x}\sin(7x)\,dx &= \frac{6}{49}e^{6x}\sin(7x) - \frac{1}{7}e^{6x}\cos(7x) \\
\int e^{6x}\sin(7x)\,dx &= \frac{6}{85}e^{6x}\sin(7x) - \frac{7}{85}e^{6x}\cos(7x) + C
\end{aligned}$$

(7) Evaluate the indefinite integral

$$\int x\sin^2(8x)\,dx$$

Solutions:

$$\begin{aligned}
&\int x\sin^2(8x)\,dx \\
&= \int \frac{1}{2}\left(x - x\cos(16x)\right)\,dx \\
&= \frac{x^2}{4} - \int \frac{1}{32}x\,d\sin(16x) \\
&= \frac{x^2}{4} - \frac{1}{32}x\sin(16x) + \int \frac{1}{32}\sin(16x)\,dx \\
&= \frac{x^2}{4} - \frac{1}{32}x\sin(16x) - \frac{1}{512}\cos(16x) + C
\end{aligned}$$

(8) Evaluate the indefinite integral

$$\int x\arctan(2x)\,dx$$

Solutions:

$$
\begin{aligned}
&\int x \arctan(2x)\, dx \\
&= \int \arctan(2x)\, d\frac{x^2}{2} \\
&= \frac{x^2}{2}\arctan(2x) - \int \frac{x^2}{2}\, d\arctan(2x) \\
&= \frac{x^2}{2}\arctan(2x) - \int \frac{2x^2}{8x^2+2}\, dx \\
&= \frac{x^2}{2}\arctan(2x) - \int \frac{1}{4} - \frac{1}{2(8x^2+2)}\, dx \\
&= \frac{x^2}{2}\arctan(2x) - \frac{x}{4} + \int \frac{1}{4(4x^2+1)}\, dx
\end{aligned}
$$

For $\int \frac{1}{4(4x^2+1)} dx$, sub $x = \frac{1}{2}\tan(\theta)$, then $dx = \frac{1}{2}\sec^2(\theta)\, d\theta$

$$
\begin{aligned}
&\frac{x^2}{2}\arctan(2x) - \frac{x}{4} + \int \frac{1}{4(4x^2+1)}\, dx \\
&= \frac{x^2}{2}\arctan(2x) - \frac{x}{4} + \int \frac{\sec^2(\theta)}{8(\tan^2(\theta)+1)}\, d\theta \\
&= \frac{x^2}{2}\arctan(2x) - \frac{x}{4} + \frac{\theta}{8} + C \\
&= \frac{x^2}{2}\arctan(2x) - \frac{x}{4} + \frac{1}{8}\arctan(2x) + C
\end{aligned}
$$

(9) Evaluate the indefinite integral

$$\int \sin^3(9x)\cos^{10}(9x)\, dx$$

Solutions:

$$
\begin{aligned}
&\int \sin^3(9x)\cos^{10}(9x)\, dx \\
&= \int \frac{-1}{9}(1-\cos^2(9x))\cos^{10}(9x)\, d\cos(9x) \\
&= \int \frac{-1}{9}\cos^{10}(9x) + \frac{1}{9}\cos^{12}(9x)\, d\cos(9x) \\
&= \frac{1}{117}\cos^{13}(9x) - \frac{1}{99}\cos^{11}(9x) + C
\end{aligned}
$$

(10) Find the following indefinite integrals

$$\int \sec^4\left(\frac{x}{2}\right)\, dx$$

Solutions:

$$\begin{aligned}
&\int \sec^4\left(\frac{x}{2}\right)\,dx\\
&= \int 2\sec^2\left(\frac{x}{2}\right)\,d\tan\left(\frac{x}{2}\right)\\
&= \int 2 + 2\tan^2\left(\frac{x}{2}\right)\,d\tan\left(\frac{x}{2}\right)\\
&= 2\tan\left(\frac{x}{2}\right) + \frac{2}{3}\tan^3\left(\frac{x}{2}\right) + C
\end{aligned}$$

(11) Evaluate the integral

$$\int \arcsin(9x)\,dx$$

Solutions:

$$\begin{aligned}
&\int \arcsin(9x)\,dx\\
&= x\arcsin(9x) - \int x\,d\arcsin(9x)\\
&= x\arcsin(9x) - \int \frac{9x}{\sqrt{1-81x^2}}\,dx\\
&= x\arcsin(9x) + \int \frac{1}{18\sqrt{1-81x^2}}\,d(1-81x^2)\\
&= x\arcsin(9x) + \frac{\sqrt{1-81x^2}}{9} + C
\end{aligned}$$

(12) Use the substitution $x = 8\tan(\theta)$ to evaluate the indefinite integral

$$\int \frac{82}{x^2\sqrt{x^2+64}}\,dx$$

Solutions:
Sub $x = 8\tan(\theta)$, then $dx = 8\sec^2(\theta)\,d\theta$.
Note that $\csc(\theta) = \dfrac{\sqrt{x^2+64}}{x}$.

$$\begin{aligned}
&\int \frac{82}{x^2\sqrt{x^2+64}}\,dx\\
&= \int \frac{82(8\sec^2(\theta))}{64\tan^2(\theta)\sqrt{64\tan^2(\theta)+64}}\,d\theta\\
&= \int \frac{41}{32}\csc(\theta)\cot(\theta)\,d\theta\\
&= -\frac{41}{32}\csc(\theta) + C\\
&= -\frac{41\sqrt{x^2+64}}{32x} + C
\end{aligned}$$

(13) Evaluate the indefinite integral

$$\int \sqrt{24x - x^2}\, dx$$

Solutions:

$$\begin{aligned} &\int \sqrt{24x - x^2}\, dx \\ = &\int \sqrt{144 - (x-12)^2}\, dx \end{aligned}$$

Sub $x - 12 = 12\sin(\theta)$. Then $dx = 12\cos(\theta)\, d\theta$

$$\begin{aligned} &\int \sqrt{144 - (x-12)^2}\, dx \\ = &\int \sqrt{144 - 144\sin^2(\theta)} \cdot 12\cos(\theta)\, d\theta \\ = &\int 144\cos^2(\theta)\, d\theta \\ = &\int 72 + 72\cos(2\theta)\, d\theta \\ = &\, 72\theta + 36\sin(2\theta) + C \end{aligned}$$

Note that $\theta = \arcsin\left(\frac{x}{12} - 1\right)$, and $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{x}{12} - 1\right)\left(\sqrt{1 - \left(\frac{x}{12} - 1\right)^2}\right)$
Then

$$\begin{aligned} &72\theta + 36\sin(2\theta) + C \\ = &\, 72\arcsin\left(\frac{x}{12} - 1\right) + 72\left(\frac{x}{12} - 1\right)\left(\sqrt{1 - \left(\frac{x}{12} - 1\right)^2}\right) + C \\ = &\, 72\arcsin\left(\frac{x}{12} - 1\right) + \frac{1}{2}(x-12)\sqrt{24x - x^2} + C \end{aligned}$$

(14) Evaluate the integral

$$\int \frac{-2t^5}{\sqrt{t^2+2}}\, dt$$

Solutions:
Sub $u = t^2 + 2$, then $du = 2t\,dt$

$$\begin{aligned}
&\int \frac{-2t^5}{\sqrt{t^2+2}}\,dt \\
&= \int \frac{-t^4}{\sqrt{u}}\,du \\
&= \int \frac{-(u-2)^2}{\sqrt{u}}\,du \\
&= \int -u^{3/2} + 4u^{1/2} - 4u^{-1/2}\,du \\
&= -\frac{2}{5}u^{5/2} + \frac{8}{3}u^{3/2} - 8u^{1/2} + C \\
&= -\frac{2}{5}(t^2+2)^{5/2} + \frac{8}{3}(t^2+2)^{3/2} - 8(t^2+2)^{1/2} + C
\end{aligned}$$

(15) Use the formula

$$\int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

and the reduction formula

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

to evaluate the integral

$$\int \sin^4(x)\,dx$$

Solutions:

$$\begin{aligned}
&\int \sin^4(x)\,dx \\
&= -\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\int \sin^2(x)\,dx \\
&= -\frac{1}{4}\cos(x)\sin^3(x) + \frac{3}{4}\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right) + C \\
&= -\frac{1}{4}\cos(x)\sin^3(x) + \frac{3x}{8} - \frac{3\sin(2x)}{16} + C
\end{aligned}$$

(16) The form of the partial fraction decomposition of a rational function is given below.

$$\frac{-(3x^2+5x+16)}{(x-5)(x^2+4)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+4}$$

Now evaluate the integral

$$\int \frac{-(3x^2+5x+16)}{(x-5)(x^2+4)}$$

Solutions:

$$\frac{-(3x^2+5x+16)}{(x-5)(x^2+4)} = \frac{A(x^2+4)+(Bx+C)(x-5)}{(x-5)(x^2+4)}$$
$$= \frac{(A+B)x^2+(C-5B)x+(4A-5C)}{(x-5)(x^2+4)}$$

Then $A=-4, B=1, C=0$

$$\int \frac{-(3x^2+5x+16)}{(x-5)(x^2+4)}\,dx$$
$$= \int -\frac{4}{x-5} + \frac{x}{x^2+4}\,dx$$
$$= -4\ln|x-5| + \frac{\ln(x^2+4)}{2} + C$$

(17) Evaluate the integral

$$\int \frac{3}{x^2+10x+25}\,dx$$

Solutions:

$$\int \frac{3}{x^2+10x+25}\,dx$$
$$= \int \frac{3}{(x+5)^2}\,d(x+5)$$
$$= -\frac{3}{x+5} + C$$

(18) Evaluate the integral

$$\int \frac{x^3+7}{x^2+5x+4}\,dx$$

Solutions:

$$\int \frac{x^3+7}{x^2+5x+4}\,dx$$
$$= \int x-5+\frac{21x+27}{(x+4)(x+1)}\,dx$$

Suppose $\frac{21x+27}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1}$, where A, B are constants. Then

$$\frac{21x+27}{(x+4)(x+1)} = \frac{A(x+1)+B(x+4)}{(x+1)(x+4)}$$
$$= \frac{(A+B)x+(A+4B)}{(x+1)(x+4)}$$

Then $A = 19, B = 2$

$$\begin{aligned}&\int x - 5 + \frac{21x+27}{(x+4)(x+1)}\,dx\\ =&\int x - 5 + \frac{19}{x+4} + \frac{4}{x+1}\,dx\\ =&\frac{x^2}{2} - 5x + 19\ln|x+4| + 2\ln|x+1| + C\end{aligned}$$

(19) Evaluate the integral

$$\int \frac{-7}{(x+a)(x+b)}\,dx$$

for the case where $a = b$ and where $a \neq b$.
Solutions:
When $a = b$,

$$\begin{aligned}&\int \frac{-7}{(x+a)^2}\,dx\\ =&\frac{7}{x+a} + C\end{aligned}$$

When $a \neq b$,
Suppose $\dfrac{-7}{(x+a)(x+b)} = \dfrac{A}{x+a} + \dfrac{B}{x+b}$, where A, B are constants.
Then

$$\begin{aligned}\frac{-7}{(x+a)(x+b)} &= \frac{A(x+b)+B(x+a)}{(x+a)(x+b)}\\ &= \frac{(A+B)x + (Ab+Ba)}{(x+a)(x+b)}\end{aligned}$$

Then $A = \dfrac{7}{a-b}, B = \dfrac{-7}{a-b}$

$$\begin{aligned}&\int \frac{-7}{(x+a)(x+b)}\,dx\\ =&\int \frac{7}{(a-b)(x+a)} - \frac{7}{(a-b)(x+b)}\,dx\\ =&\frac{7}{a-b}(\ln|x+a| - \ln|x+b|) + C\end{aligned}$$

(20) Evaluate the integral

$$\int \frac{10x}{x^4 - a^4}\,dx$$

Solutions:
(Solution 1)

Sub $x^2 = a^2\sec(\theta), dx = \dfrac{a^2\sec(\theta)\tan(\theta)}{2x}\,d\theta$
Note that $\csc(\theta) = \dfrac{x^2}{\sqrt{x^4-a^4}}, \cot(\theta) = \dfrac{a^2}{\sqrt{x^4-a^4}}$

$$\begin{aligned}
&\int \frac{10x}{x^4-a^4}\,dx \\
=&\int \frac{5a^2\sec(\theta)\tan(\theta)}{a^4\sec^2(\theta)-a^4}\,d\theta \\
=&\int \frac{5}{a^2}\csc(\theta)\,d\theta
\end{aligned}$$

Sub $u = \csc(\theta)+\cot(\theta)$, then $du = -\csc(\theta)\cot(\theta) - \csc^2(\theta)\,d\theta = -u\csc(\theta)\,d\theta$.

$$\begin{aligned}
&\int \frac{5}{a^2}\csc(\theta)\,d\theta \\
=&\int \frac{5}{a^2}\csc(\theta)\cdot\frac{-1}{u\csc(\theta)}\,du \\
=&\int -\frac{5}{a^2u}\,du \\
=&-\frac{5}{a^2}\ln|u| + C \\
=&-\frac{5}{a^2}\ln|\csc(\theta)+\cot(\theta)| + C \\
=&-\frac{5}{a^2}\ln\left|\frac{x^2+a^2}{\sqrt{x^4-a^4}}\right| + C \\
=&\frac{5}{a^2}\ln\left|\frac{\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}}\right| + C \\
=&\frac{5}{2a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C
\end{aligned}$$

(Solution 2)
Suppose $\dfrac{10x}{x^4-a^4} = \dfrac{Ax+B}{x^2+a^2} + \dfrac{C}{x+a} + \dfrac{D}{x-a}$, where A, B, C, D are constants. Then

$$\begin{aligned}
\frac{10x}{x^4-a^4} &= \frac{(Ax+B)(x^2-a^2)+C(x^3-ax^2+a^2x-a^3)+D(x^3+ax^2+a^2x+a^3)}{(x^2+a^2)(x+a)(x-a)} \\
&= \frac{(A+C+D)x^3+(B+Da-Ca)x^2+(Ca^2+Da^2-Aa^2)x+(Da^3-Ba^2-Ca^3)}{(x^2+a^2)(x+a)(x-a)}
\end{aligned}$$

Then $A = \frac{-5}{a^2}, B = 0, C = \frac{5}{2a^2}, D = \frac{5}{2a^2}$

$$\begin{aligned}
&\int \frac{10x}{x^4 - a^4}\,dx \\
&= \frac{1}{a^2}\int \frac{-5x}{(x^2+a^2)} + \frac{5}{2(x+a)} + \frac{5}{2(x-a)}\,dx \\
&= \frac{1}{a^2}\left(-\frac{5}{2}\ln|x^2+a^2| + \frac{5}{2}\ln|x+a| + \frac{5}{2}\ln|x-a|\right) + C \\
&= \frac{5}{2a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C
\end{aligned}$$

(21) Find the integral

$$\int y\sqrt{y+1}\,dy$$

Solutions:

$$\begin{aligned}
&\int y\sqrt{y+1}\,dy \\
&= \int (y+1)\sqrt{y+1} - \sqrt{y+1}\,dy \\
&= \int (y+1)^{3/2} - (y+1)^{1/2}\,dy \\
&= \frac{2}{5}(y+1)^{5/2} - \frac{2}{3}(y+1)^{3/2} + C
\end{aligned}$$

(22) Evaluate the integral

$$\int -7\sqrt{3-2x-x^2}\,dx$$

Solutions:

$$\begin{aligned}
&\int -7\sqrt{3-2x-x^2}\,dx \\
&= \int -7\sqrt{4-(x+1)^2}\,dx
\end{aligned}$$

Sub $x+1=2\sin(\theta)$. Then $dx = 2\cos(\theta)d\theta$

$$\begin{aligned}
&\int -7\sqrt{4-(x+1)^2}\,dx\\
&=\int -7\sqrt{4-4\sin^2(\theta)}\cdot 2\cos(\theta)\,d\theta\\
&=\int -28\cos^2(\theta)\,d\theta\\
&=\int -14-14\cos(2\theta)\,d\theta\\
&=-14\theta-7\sin(2\theta)+C
\end{aligned}$$

Note that $\theta = \arcsin\left(\frac{x+1}{2}\right)$, and $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{x+1}{2}\right)\left(\sqrt{1-\left(\frac{x+1}{2}\right)^2}\right)$

Then

$$\begin{aligned}
&-14\theta-7\sin(2\theta)+C\\
&=-14\arcsin\left(\frac{x+1}{2}\right)-14\left(\frac{x+1}{2}\right)\left(\sqrt{1-\left(\frac{x+1}{2}\right)^2}\right)+C\\
&=-14\arcsin\left(\frac{x+1}{2}\right)-\frac{7}{2}(x+1)\sqrt{3-2x-x^2}+C
\end{aligned}$$

(23) Evaluate the integral when $x > 0$

$$\int \ln(x^2+11x+18)\,dx$$

Solutions:

$$\begin{aligned}
&\int \ln(x^2+11x+18)\,dx\\
&=\int \ln(x+2)+\ln(x+9)\,dx\\
&=(x+2)\ln(x+2)+(x+9)\ln(x+9)-\int(x+2)\,d\ln(x+2)-\int(x+9)\,d\ln(x+9)\\
&=(x+2)\ln(x+2)+(x+9)\ln(x+9)-2x+C
\end{aligned}$$

(24) Evaluate the integral

$$\int \frac{-9}{x+4+4\sqrt{x+1}}\,dx$$

Solutions:

Sub $u = \sqrt{x+1}, du = \dfrac{1}{2\sqrt{x+1}}\,dx = \dfrac{1}{2u}\,dx$

$$\begin{aligned}
&\int \frac{-9}{x+4+4\sqrt{x+1}}\,dx \\
&= \int \frac{-18u}{(u+3)(u+1)}\,du
\end{aligned}$$

Let $\dfrac{-18u}{(u+3)(u+1)} = \dfrac{A}{u+3} + \dfrac{B}{u+1}$, where A, B are constants. Then

$$\begin{aligned}
\frac{-18u}{(u+3)(u+1)} &= \frac{A(u+1)+B(u+3)}{(u+3)(u+1)} \\
&= \frac{(A+B)u+(A+3B)}{(u+3)(u+1)}
\end{aligned}$$

Then $A = -27, B = 9$.

$$\begin{aligned}
&\int \frac{-18u}{(u+3)(u+1)}\,du \\
&= \int \frac{-27}{u+3} + \frac{9}{u+1}\,du \\
&= -27\ln|u+3| + 9\ln|u+1| + C \\
&= -27\ln(\sqrt{x+1}+3) + 9\ln(\sqrt{x+1}+1) + C
\end{aligned}$$