THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Coursework 8

If you find any errors or typos, please email us at
math1010@math.cuhk.edu.hk

(1) Suppose that $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function f about $x = 0$. What can you say about the signs of a, b, c if f has the graph given below? .

Solution:

From the figure, $f(0) < 0, f'(0) < 0$ and $f''(0) > 0$. Hence, $a = f(0) < 0$, $b = f'(0) < 0$ and $c = \frac{f''(0)}{2!} > 0$.

(2) Find the first four Taylor polynomials about $x = x_0$.

$\ln(x + 9); x_0 = -8$

Solution:

$f(x) = \ln(x + 9)$.

$f(-8) = \ln(1) = 0$.

$f'(x) = \frac{1}{x+9}$

$f'(-8) = 1$

$f''(x) = -\frac{1}{(x+9)^2}$

$f''(-8) = -1$

$f^{(3)}(x) = \frac{2}{(x+9)^3}$

$f^{(3)}(-8) = 2$

Then we have

$$\begin{aligned} p_3(x) &= 0 + (x + 8) + \frac{-1}{2!}(x + 8)^2 + \frac{2}{3!}(x + 8)^3 \\ &= (x + 8) - \frac{1}{2}(x + 8)^2 + \frac{1}{3}(x + 8)^3 \end{aligned}$$

$$p_2(x) = (x + 8) - \frac{1}{2}(x + 8)^2$$

$$p_1(x) = x + 8$$

$$p_0(x) = 0$$

(3) Find the first three **nonzero** terms of the Taylor series for the function $f(x) = \sqrt{10x - x^2}$ about the point $a = 5$.

Solution:

$f(5) = 5$

$f'(x) = \dfrac{10 - 2x}{2\sqrt{10x - x^2}} = \dfrac{5 - x}{\sqrt{10x - x^2}}$

$f'(5) = 0$

$f''(x) = -\dfrac{25}{(10x - x^2)^{\frac{3}{2}}}$

$f''(5) = -\dfrac{1}{5}$

$f^{(3)}(x) = -\dfrac{75(x - 5)}{(10x - x^2)^{\frac{5}{2}}}$

$f^{(3)}(5) = 0$

$f^{(4)}(x) = -\dfrac{75(4x^2 - 40x + 125)}{(10x - x^2)^{\frac{7}{2}}}$

$f^{(4)}(5) = -\dfrac{3}{125}$

Then

$$\sqrt{10x-x^2} = 5 + \frac{1}{2!}(-\frac{1}{5})(x-5)^2 + \frac{1}{4!}(-\frac{3}{125})(x-5)^4 + ...$$
$$= 5 - \frac{1}{10}(x-5)^2 - \frac{1}{1000}(x-5)^4 + ...$$

(4) **Taylor and Maclaurin Series**: Compute the Taylor Series below.

$e^x =$__$+$__$x+$__x^2+__x^3+__$x^4+...$

$\cos^2 x =$__$+$__$x+$__x^2+__x^3+__$x^4+...$

$x^x =$__$+$__$(x-1)+$__$(x-1)^2+$__$(x-1)^3+$__$(x-1)^4+...$

$4x^4+3x^3+2x^2+x+1 =$__$+$__$(x-1)+$__$(x-1)^2+$__$(x-1)^3+$__$(x-1)^4$

Solution:

For convenience, denote $f(x) = e^x$, $g(x) = \cos^2 x$, $h(x) = x^x$ and $k(x) = 4x^4 + 3x^3 + 2x^2 + x + 1$.

$f^{(n)}(x) = e^x$ for all n. Therefore, $f^{(n)}(0) = 1$ for all n. Hence, we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...$$
$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + ...$$

Write $g(x) = \frac{1}{2}(\cos(2x)+1)$. Then $g'(x) = -\sin(2x), g''(x) = -2\cos(2x), g^{(3)}(x) = 4\sin(2x), g^{(4)}(x) = 8\cos(2x)$. Therefore, $g(0) = 1, g'(0) = 0, g''(0) = -2, g^{(3)}(0) = 0, g^{(4)}(0) = 8$. Then we have

$$\cos^2 x = 1 + \frac{1}{2!}(-2)x^2 + \frac{1}{4!}(8)x^4 + ...$$
$$= 1 - x^2 + \frac{1}{3}x^4 + ...$$

Write $h(x) = e^{x\ln x}$. Then $h'(x) = x^x(\ln x+1), h''(x) = x^{(x-1)}(x+x(\ln x)^2+2x\ln x+1), h^{(3)}(x) = x^{(x-2)}(x^2+x^2(\ln x)^3+3x^2(\ln x)^2+3x+3x(x+1)\ln x+1), h^{(4)}(x) = x^{(x-3)}(x^3+x^3(\ln x)^4+4x^3(\ln x)^3+6x^2+6x^2(x+1)(\ln x)^2+4x(x^2+3x-1)\ln x-x+2$. Therefore, $h(1) = 1, h'(1) = 1, h''(1) = 2, h^{(3)}(1) = 3, h^{(4)}(1) = 8$. Then we have

$$x^x = 1 + (x-1) + \frac{1}{2!}(2)(x-1)^2 + \frac{1}{3!}(3)(x-1)^3 + \frac{1}{4!}(8)(x-1)^4 + ...$$
$$= 1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{3}(x-1)^4 + ...$$

$k'(x) = 16x^3+9x^2+4x+1, k''(x) = 48x^2+18x+4, k^{(3)}(x) = 96x+18, k^{(4)}(x) = 96$. Therefore, $k(1) = 11, k'(1) = 30, k''(1) = 70, k^{(3)}(1) = 114, k^{(4)}(1) = 96$. Then we have

$$4x^4 + 3x^3 + 2x^2 + x + 1$$
$$= 11 + 30(x-1) + \frac{1}{2!}(70)(x-1)^2 + \frac{1}{3!}(114)(x-1)^3 + \frac{1}{4!}(96)(x-1)^4$$
$$= 11 + 30(x-1) + 35(x-1)^2 + 19(x-1)^3 + 4(x-1)^4$$

(5) Suppose that $f(x)$ and $g(x)$ are given by the power series

$$f(x) = 5 + 3x + 5x^2 + 2x^3 + ...$$

and

$$g(x) = 3 + 3x + 3x^2 + 4x^3 + ...$$

By multiplying power series, find the first few terms of the series for the product

$$h(x) = f(x) \cdot g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

Solution:
$c_0 = 5 \times 3 = 15$
$c_1 = 5 \times 3 + 3 \times 3 = 24$
$c_2 = 5 \times 3 + 3 \times 3 + 5 \times 3 = 39$
$c_3 = 5 \times 4 + 3 \times 3 + 5 \times 3 + 2 \times 3 = 50$

(6) Suppose that you are told that the Taylor series of $f(x) = x^2 e^{x^2}$ about $x = 0$ is

$$x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + ...$$

Find $\frac{d}{dx}(x^2 e^{x^2})\big|_{x=0}$ and $\frac{d^6}{dx^6}(x^2 e^{x^2})\big|_{x=0}$.

Solution: The coefficient of x is 0, so $\frac{d}{dx}(x^2 e^{x^2})\big|_{x=0} = 0$.

The coefficient of x^6 is $\frac{1}{2}$, so $\frac{1}{6!}\frac{d^6}{dx^6}(x^2 e^{x^2})\big|_{x=0} = \frac{1}{2} \implies \frac{d^6}{dx^6}(x^2 e^{x^2})\big|_{x=0} = 360$

(7) Let $f(x) = \dfrac{\cos(3x^3) - 1}{x^3}$. Evaluate the 9^{th} derivative of f at $x = 0$.

Solution:
It is known that

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + ... = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ...$$

Then we have

$$\begin{aligned} f(x) &= \frac{\cos(3x^3) - 1}{x^3} \\ &= \frac{(1 - \frac{1}{2}(3x^3)^2 + \frac{1}{24}(3x^3)^4 + ...) - 1}{x^3} \\ &= \frac{-\frac{9}{2}x^6 + \frac{27}{8}x^{12} + ...}{x^3} \\ &= -\frac{9}{2}x^3 + \frac{27}{8}x^9 + ... \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{1}{9!}f^{(9)}(0) &= \frac{27}{8} \\ f^{(9)}(0) &= 1224720 \end{aligned}$$

(8) (a) By considering Taylor polynomials, find the local quadratic approximation of $\sqrt{\frac{x}{2}}$ at $x_0 = 2$.

(b) Use the result obtained in part (a) to approximate $\sqrt{1.06}$.

Solution:

(a) $f(x) = \sqrt{\frac{x}{2}}$.

$f(2) = 1$

$f'(x) = \frac{1}{2\sqrt{2}\sqrt{x}}$

$f'(2) = \frac{1}{4}$

$f''(x) = -\frac{1}{4\sqrt{2}x^{\frac{3}{2}}}$

$f''(2) = -\frac{1}{16}$

Then the quadratic approximation formula is

$$\sqrt{\frac{x}{2}} \approx 1 + \frac{1}{4}(x-2) + \frac{1}{2!}(-\frac{1}{16})(x-2)^2 = 1 + \frac{1}{4}(x-2) - \frac{1}{32}(x-2)^2$$

(b) Substitute $x = \frac{53}{25}$ into the approximation in (a).

$$\begin{aligned}\sqrt{1.06} &\approx 1 + \frac{1}{4}(\frac{53}{25} - 2) - \frac{1}{32}(\frac{53}{25} - 2)^2 \\ &= 1.02955\end{aligned}$$

(9) Find the Taylor polynomial of degree 4 for $\cos(x)$, for $x = 0$. Approximate $\cos(x)$ with $T_4(x)$ to simplify the ratio $\frac{1-\cos(x)}{x}$. Using this, conclude the limit

$$\lim_{x\to 0} \frac{1-\cos(x)}{x}$$

Solution:

$T_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Then $\frac{1-\cos(x)}{x} \approx \frac{1-(1-\frac{1}{2}x^2+\frac{1}{24}x^4)}{x} = \frac{1}{2}x - \frac{1}{24}x^3$. Therefore,

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$$

(10) Find an antiderivative P of $5\sin(5s)$.

Solution:

$$\begin{aligned}P(s) &= \int (5\sin(5s))ds \\ &= -\cos(5s) + C\end{aligned}$$

(11) Find the most general antiderivative of $f(t) = -9\cos(t) - 10\sin(t)$.
Solution:
The required antiderivative is given by

$$\int (-9\cos(t) - 10\sin(t))dt = -9\sin(t) + 10\cos(t) + C$$

(12) Evaluate the following indefinite integral

$$\int (6\sin(x) - 5\cos(x))dx$$

Solution:

$$\int (6\sin(x) - 5\cos(x))dx = -6\cos(x) - 5\sin(x) + C$$

(13) Calculate the following antiderivatives:
(a)

$$\int \frac{11}{x}dx$$

(b)

$$\int (-3\sin x + 8\cos x)dx$$

(c)

$$\int -4e^x dx$$

Solution:
(a)

$$\int \frac{11}{x}dx = 11\ln|x| + C$$

(b)

$$\int (-3\sin x + 8\cos x)dx = 3\cos x + 8\sin x + C$$

(c)

$$\int -4e^x dx = -4e^x + C$$

(14) Find the general indefinite integral:

$$\int (-x^2 - 6 + \frac{-2}{x^2+1})dx$$

Solution:

$$\int (-x^2 - 6 + \frac{-2}{x^2+1})dx = -\frac{x^3}{3} - 6x - 2\arctan(x) + C$$

(15) Evaluate the indefinie integral

$$\int (8\sin(t) + \cos(t) - 5\sec^2(t) + 4e^t + \frac{3}{\sqrt{1-t^2}} + \frac{2}{1+t^2})dt$$

Solution:

$$\int (8\sin(t) + \cos(t) - 5\sec^2(t) + 4e^t + \frac{3}{\sqrt{1-t^2}} + \frac{2}{1+t^2})dt$$
$$= -8\cos(t) + \sin(t) - 5\tan(t) + 4e^t + 3\arcsin(t) + 2\arctan(t) + C$$

(16) Given that $f''(x) = \cos(x)$, $f'(\pi/2) = 11$ and $f(\pi/2) = 12$, find $f'(x)$ and $f(x)$.
Solution:

$$\begin{aligned} f'(x) &= \int \cos(x)dx \\ &= \sin(x) + C_1 \end{aligned}$$

$f'(\pi/2) = 11 \implies 11 = \sin(\pi/2) + C_1 \implies C_1 = 10$
Hence,

$$f'(x) = \sin(x) + 10$$

$$\begin{aligned} f(x) &= \int (\sin(x) + 10)dx \\ &= -\cos(x) + 10x + C_2 \end{aligned}$$

$f(\pi/2) = 12 \implies 12 = -\cos(\pi/2) + 10(\frac{\pi}{2}) + C_2 \implies C_2 = 12 - 5\pi.$
Hence,

$$f(x) = -\cos(x) + 10x + 12 - 5\pi$$

(17) Consider the function $f(x)$ whose second derivative is $f''(x) = 9x + 9\sin(x)$. If $f(0) = 3$ and $f'(0) = 2$, what is $f(x)$?
Solution:

$$\begin{aligned} f'(x) &= \int (9x + 9\sin(x))dx \\ &= \frac{9x^2}{2} - 9\cos(x) + C_1 \end{aligned}$$

$f'(0) = 2 \implies 2 = -9 + C_1 \implies C_1 = 11$
Hence,

$$f'(x) = \frac{9x^2}{2} - 9\cos(x) + 11$$

$$\begin{aligned} f(x) &= \int (\frac{9x^2}{2} - 9\cos(x) + 11)dx \\ &= \frac{3x^3}{2} - 9\sin(x) + 11x + C_2 \end{aligned}$$

$f(0) = 3 \implies 3 = C_2$
Hence,

$$f(x) = \frac{3x^3}{2} - 9\sin(x) + 11x + 3$$

(18) Solve the following initial value problem.

$$\frac{dy}{dx} = x\sqrt{x^7}, y(0) = 0$$

Solution:

$$\begin{aligned} y &= \int (x\sqrt{x^7})dx \\ &= \int x^{9/2}dx \\ &= \frac{2}{11}x^{\frac{11}{2}} + C \end{aligned}$$

$y(0) = 0 \implies C = 0$
Hence,

$$y = \frac{2}{11}x^{\frac{11}{2}}$$