THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Suggested Solution of Coursework 7

If you find any errors or typos, please email us at
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1. (1 point) Determine the intervals on which the given function is concave up or down and find the point of inflection. Let

$$f(x) = x\left(x - 6\sqrt{x}\right)$$

The x-coordinate of the point of inflection is ______

The interval on the left of the inflection point is ______, and on this interval f is [?].

The interval on the right is ______, and on this interval f is [?].

Solution: Notice that $f(x)$ is twice differentiable on its domain except at $x = 0$. We compute the first derivative and the second derivative of $f(x)$:

$$f'(x) = \left(x - 6\sqrt{x}\right) + x\left(1 - \frac{3}{\sqrt{x}}\right) = 2x - 9\sqrt{x}$$

$$f''(x) = 2 - \frac{9}{2\sqrt{x}}$$

We can see that $f''(x) = 0$ if and only if $x = \frac{81}{16}$.

Since $f''(x) < 0$ when $x \in \left(0, \frac{81}{16}\right)$ and $f''(x) > 0$ when $x \in \left(\frac{81}{16}, \infty\right)$, we can see that $f(x)$ has a unique inflection point at $x = \frac{81}{16}$.

On the left interval $\left(0, \frac{81}{16}\right)$, $f''(x) < 0$, so $f(x)$ is concave down on this interval.

On the right interval $\left(\frac{81}{16}, \infty\right)$, $f''(x) > 0$, so $f(x)$ is concave up on this interval.

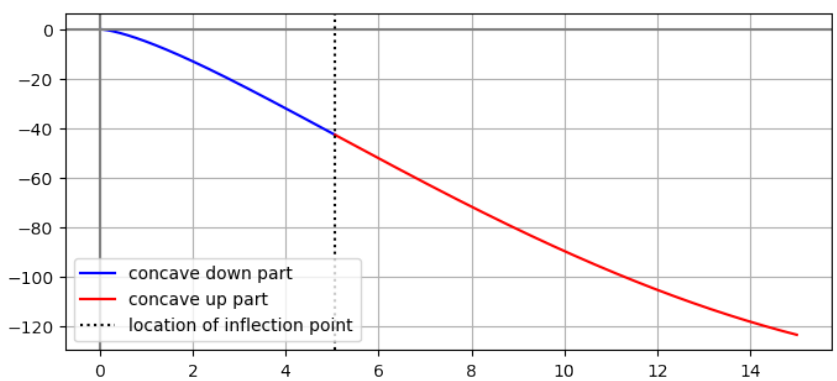


Figure 1: The graph of $f(x) = x(x - 6\sqrt{x})$

2. (1 point) Answer the following questions for the function

$$f(x) = \frac{x^3}{x^2 - 1}$$

defined on the interval $[-17, 17]$.

a.) Enter the x-coordinates of the vertical asymptotes of $f(x)$ as a comma-separated list. That is, if there is just one value, give it; if there are more than one, enter them separated commas; and if there are none, enter *NONE* .

b.) $f(x)$ is concave up on the region ______________.

c.) Enter the x-coordinates of the inflection point(s) for this function as a comma-separated list.

Solution: First note that the function is not defined on $x = 1$ and $x = -1$, and is not differentiable at $x = -17$ and $x = 17$ as only one of the one-sided limits is defined.

a.) Vertical asymptotes occur when the denominator $x^2 - 1$ is 0, which happens when $x = 1$ or $x = -1$.

b.) We compute the second derivative

$$\begin{aligned} f''(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{3x^2(x^2-1) - x^3 \cdot 2x}{(x^2-1)^2} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{x^4 - 3x^2}{(x^2-1)^2} \\ &= \frac{(4x^3 - 6x)(x^2-1)^2 - (x^4 - 3x^2) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{2x(x^2+3)}{(x^2-1)^3} \end{aligned}$$

The function is concave up on where $f''(x) > 0$. Since $2(x^2 + 3) > 0$ on the domain, $f''(x) > 0$ if and only if $\frac{x}{(x^2-1)^3} > 0$, which holds if and only if $x \in (-1, 0) \cup (1, 17)$. This means that $f(x)$ is concave up on $(-1, 0)$ and on $(1, 17)$ respectively.

c.) $f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3} = 0$ if and only if $x = 0$. Since near $x = 0$, $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, there is an inflection point at $x = 0$.

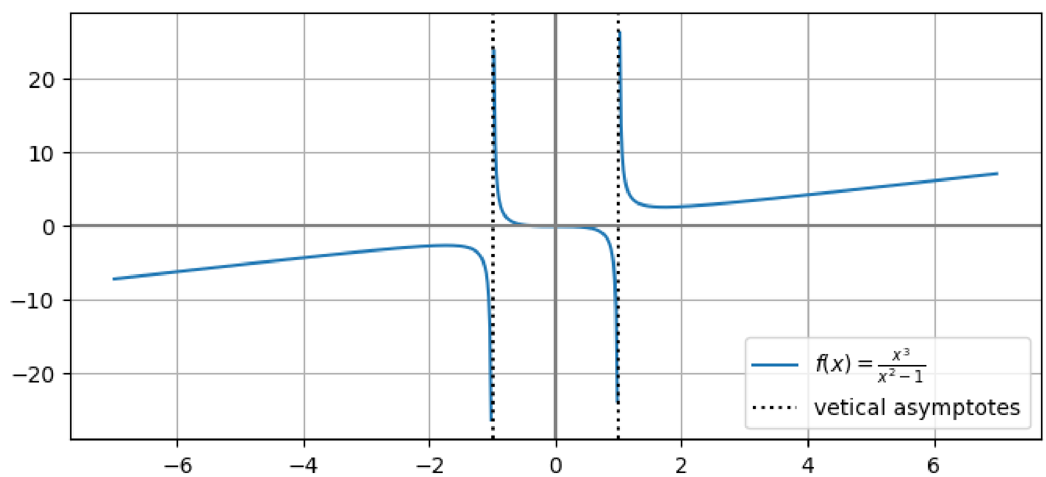
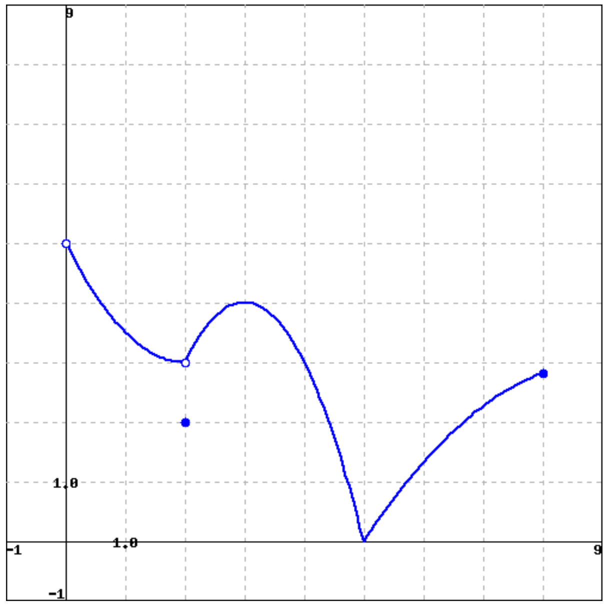


Figure 2: The graph of $f(x) = \frac{x^3}{x^2-1}$

3. (1 point) Use the given graph of the function on the interval $(0, 8]$ to answer the following questions.



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1. Where does the function f have a local maximum?
Answer (separate by commas): $x =$ __________

2. Where does the function f have a local minimum?
Answer (separate by commas): $x =$ __________

3. What is the global maximum of f?
Answer (write 'none' if there is none): ________

4. What is the global minimum of f?
Answer (write 'none' if there is none): ________

Solution: **1.** As shown in the graph, $x = 3$ and $x = 8$ are two local maxima. Note that $x = 0$ is not a local maximum as it is not included in the domain.

2. As shown in the graph, $x = 2$ and $x = 5$ are two local minima.

3. The function has no global maximum as for every point in $(0, 8]$ there is one point near 0 that has larger value.

4. As shown in the graph, $x = 5$ is the global minimum.

4. (1 point) Please answer the following questions about the function

$$f(x) = \frac{5x^2}{x^2 - 9}.$$

(a) Calculate the first derivative of f. Find the critical numbers of f, where it is increasing and decreasing, and its local extrema.

$f'(x) =$ ____________________
Critical numbers $x =$ ____________________
Union of the intervals where $f(x)$ is increasing ____________________
Union of the intervals where $f(x)$ is decreasing ____________________
Local maxima $x =$ ____________________
Local minima $x =$ ____________________

(b) Find the following left- and right-hand limits at the vertical asymptote $x = -3$.

$$\lim_{x \to -3^-} \frac{5x^2}{x^2 - 9} = \boxed{?} \qquad \lim_{x \to -3^+} \frac{5x^2}{x^2 - 9} = \boxed{?}$$

Find the following left- and right-hand limits at the vertical asymptote $x = 3$.

$$\lim_{x \to 3^-} \frac{5x^2}{x^2 - 9} = \boxed{?} \qquad \lim_{x \to 3^+} \frac{5x^2}{x^2 - 9} = \boxed{?}$$

Find the following limits at infinity to determine any horizontal asymptotes.

$$\lim_{x \to -\infty} \frac{5x^2}{x^2 - 9} = \boxed{?} \qquad \lim_{x \to +\infty} \frac{5x^2}{x^2 - 9} = \boxed{?}$$

(c) Calculate the second derivative of f. Find where f is concave up, concave down, and has inflection points.

$f''(x) =$ ________________
Union of the intervals where $f(x)$ is concave up ________________
Union of the intervals where $f(x)$ is concave down ________________
Inflection points $x =$ ________________

(d) The function f is $\boxed{?}$ because $\boxed{?}$ for all x in the domain of f, and therefore its graph is symmetric about the $\boxed{?}$

(e) Answer the following questions about the function f and its graph.
The domain of f is the set (in **interval notation**) ________________
The range of f is the set (in **interval notation**) ________________
y-intercept ________________
x-intercepts ________________

(f) Sketch a graph of the function f without having a graphing calculator do it for you. Plot the y-intercept and the x-intercepts, if they are known. Draw dashed lines for horizontal and vertical asymptotes. Plot the points where f has local maxima, local minima, and inflection points. Use what you know from parts (a) - (c) to sketch the remaining parts of the graph of f. Use any symmetry from part (d) to your advantage. Sketching graphs is an important skill that takes practice, and you may be asked to do it on quizzes or exams.

Solution: Note that f is not defined on $x = 3$ and on $x = -3$, and is twice differentiable on the domain.

(a) The derivative of f is $f'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{5x^2}{x^2-9} = \dfrac{10x(x^2-9) - 5x^2 \cdot 2x}{(x^2-9)^2} = \dfrac{-90x}{(x^2-9)^2}$.

The critical point occurs at $f'(x) = 0$, which is only at $x = 0$.

The derivative $f'(x) = \dfrac{-90x}{(x^2-9)^2} > 0$ on $(-\infty, -3) \cup (-3, 0)$. Since $f(x)$ is continuous at the critical point $x = 0$ and it is on the boundary of $(-3, 0)$, f is increasing on $(-\infty, -3)$ and on $(-3, 0]$ respectively.

The derivative $f'(x) = \dfrac{-90x}{(x^2-9)^2} < 0$ on $(0, 3) \cup (3, \infty)$. Since $f(x)$ is continuous at the critical point $x = 0$ and it is on the boundary of $(0, 3)$, f is decreasing on $[0, 3)$ and on $(3, \infty)$ respectively.

Since f is differentiable in the domain and f has only one critical point (at $x = 0$), it suffice to check if $x = 0$ has a local extremum.
At $x = 0$, $f(0) = 0$. Also, near $x = 0$, $x^2 - 9 < 0$, so $f(x) = \frac{5x^2}{x^2-9} < 0 = f(0)$. Hence 0 is a local minimum.
As f has no other critical points and is differentiable on the domain, f has no local maxima.

(b)

1. $\lim_{x\to-3^-} \dfrac{5x^2}{x^2-9} = \lim_{x\to-3^-} \dfrac{5x^2}{(x-3)(x+3)} = -\dfrac{45}{6} \lim_{x\to-3^-} \dfrac{1}{x+3} = -\infty.$

2. $\lim_{x\to-3^+} \dfrac{5x^2}{x^2-9} = \lim_{x\to-3^+} \dfrac{5x^2}{(x-3)(x+3)} = -\dfrac{45}{6} \lim_{x\to-3^+} \dfrac{1}{x+3} = +\infty.$

3. $\lim_{x\to3^-} \dfrac{5x^2}{x^2-9} = \lim_{x\to3^-} \dfrac{5x^2}{(x-3)(x+3)} = \dfrac{45}{6} \lim_{x\to3^-} \dfrac{1}{x-3} = -\infty.$

4. $\lim_{x\to3^+} \dfrac{5x^2}{x^2-9} = \lim_{x\to3^+} \dfrac{5x^2}{(x-3)(x+3)} = \dfrac{45}{6} \lim_{x\to3^+} \dfrac{1}{x-3} = +\infty.$

5. $\lim_{x\to-\infty} \dfrac{5x^2}{x^2-9} = \lim_{x\to-\infty} \dfrac{5}{1-\frac{9}{x^2}} = 5.$

6. $\lim_{x\to+\infty} \dfrac{5x^2}{x^2-9} = \lim_{x\to+\infty} \dfrac{5}{1-\frac{9}{x^2}} = 5.$

(c)

$$\begin{aligned} f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{-90x}{(x^2-9)^2} \\ &= -90 \cdot \frac{(x^2-9)^2 - x \cdot 2(x^2-9) \cdot 2x}{(x^2-9)^4} \\ &= -90 \cdot \frac{-3(x^2+3)}{(x^2-9)^3} = \frac{270(x^2+3)}{(x^2-9)^3} \end{aligned}$$

f is concave up on where $f''(x) = \dfrac{270(x^2+3)}{(x^2-9)^3} > 0$, so it is concave up on $(-\infty, -3)$ and on $(3, \infty)$ respectively.
Similarly, f is concave down on where $f''(x) = \dfrac{270(x^2+3)}{(x^2-9)^3} < 0$, so it is concave

down on $(-3, 3)$.

$f''(x) = \dfrac{270(x^2+3)}{(x^2-9)^3} = 0$ has no solution on the domain, so f has no inflection point.

(d) As $f(-x) = \frac{5(-x)^2}{(-x)^2-9} = \frac{5x^2}{x^2-9} = f(x)$ for all x on the domain, f is an even function. Hence the graph is symmetric about the y-axis.

(e) As the function is defined on every point in $\mathbb{R}$ except for $x = 3$ and $x = -3$, the domain is $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

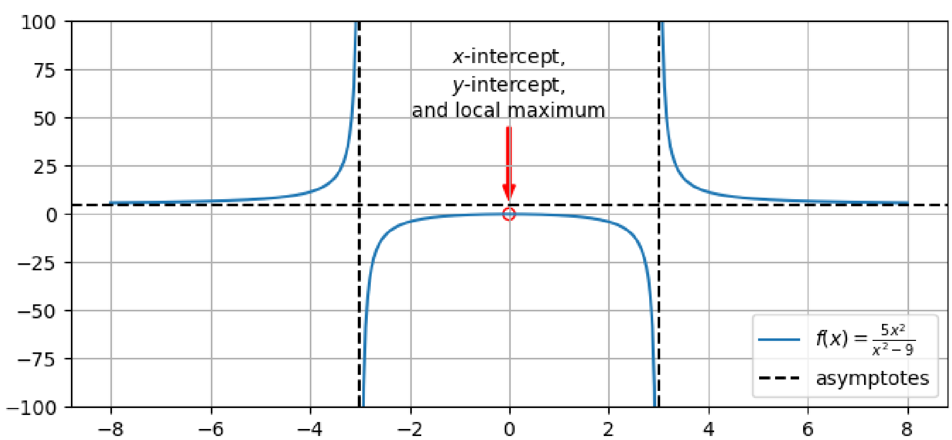
Let r be a real number. Then r is in the range if and only if there is a x in the domain such that $f(x) = \dfrac{5x^2}{x^2-9} = r$, or equivalently that the equation $(5-r)x^2 + 9r = 0$ has a solution x that is in the domain of $f(x)$.

When $x = 3$ or $x = -3$, the left hand side becomes $(5-r) \cdot 9 + 9r = 45 \neq 0$. So $3, -3$ are not solutions of the equation.
When $r = 5$, the equation becomes $0x^2 + 45 = 0$, which has no solution.
When $r \neq 5$, the discriminant of the quadratic equation is $-4(5-r) \cdot 9r = 36r(r-5)$, which is nonnegative if and only if $r \leq 0$ or $r > 5$ ($r = 5$ is eliminated).

So, this equation has a solution if and only if $r \in (-\infty, 0] \cup (5, \infty)$. Therefore the range of f is $(-\infty, 0] \cup (5, \infty)$.

The y-intercept is $f(0) = 0$.
The x-intercept is the solution of the equation $f(x) = 0$, which is 0.

(f)



5. (1 point) Suppose that

$$f(x) = (x+2)(x-6)^2.$$

(A) Find all critical values of f. If there are no critical values, enter None. If there are more than one, enter them separated by commas.
Critical value(s) = ______

(B) Use the second derivative test to find the x-coordinates of all local maxima of f. If there are no local maxima, enter None. If there are more than one, enter them separated by commas.

Local maxima at $x =$ _____

(C) Use the second derivative test to find the x-coordinates of all local minima of f. If there are no local minima, enter None. If there are more than one, enter them separated by commas.

Local minima at $x =$ _____

Solution: (A) Notice that $f(x)$ is differentiable on the domain.
We first compute the derivative

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x+2)(x-6)^2 = (x-6)^2+(x+2)\cdot 2(x-6) = 3x^2-20x+12 = (3x-2)(x-6)$$

So, the critical points are $x = \frac{2}{3}$ and $x = 6$, and their values are $f(\frac{2}{3}) = \frac{2048}{27}$ and $f(6) = 0$.

(B) The second derivative of f is $f''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 20x + 12) = 6x - 20$.
It suffices to check the second derivative of the critical points. At $x = \frac{2}{3}$, $f''(\frac{2}{3}) = -16 < 0$. At $x = 6$, $f''(6) = 16 > 0$.
Therefore the only local maxima is at $x = \frac{2}{3}$.

(C) As shown in (B), the only local minima is at $x = 6$.

6. (1 point) Let $f(x) = 2\sqrt{x} - 8x$ for $x > 0$. Find the open intervals on which f is increasing (decreasing).

   1. f is increasing on the intervals ____________
   2. f is decreasing on the intervals ____________

   Notes: In the first two, your answer should either be a single interval, such as (0,1), a comma separated list of intervals, such as (-inf, 2), (3,4), or the word "none".

Solution: Notice that $f(x)$ is differentiable on the domain.
We first compute the derivative of f:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(2\sqrt{x} - 8x) = \frac{1}{\sqrt{x}} - 8$$

From the derivative we can see that $f(x)$ has a critical point $x = \frac{1}{64}$.
f is increasing on the interval where $f'(x) > 0$ and decreasing on the interval where $f'(x) < 0$.
As $f'(x) = \frac{1}{\sqrt{x}} - 8 > 0$ if and only if $0 < x < \frac{1}{64}$ and $f'(x) < 0$ if and only if $x > \frac{1}{64}$, and from the fact that the critical point $x = \frac{1}{64}$ is at the boundary of both intervals, f is increasing on the interval $\left(0, \frac{1}{64}\right]$ and decreasing on the interval $\left[\frac{1}{64}, \infty\right)$.

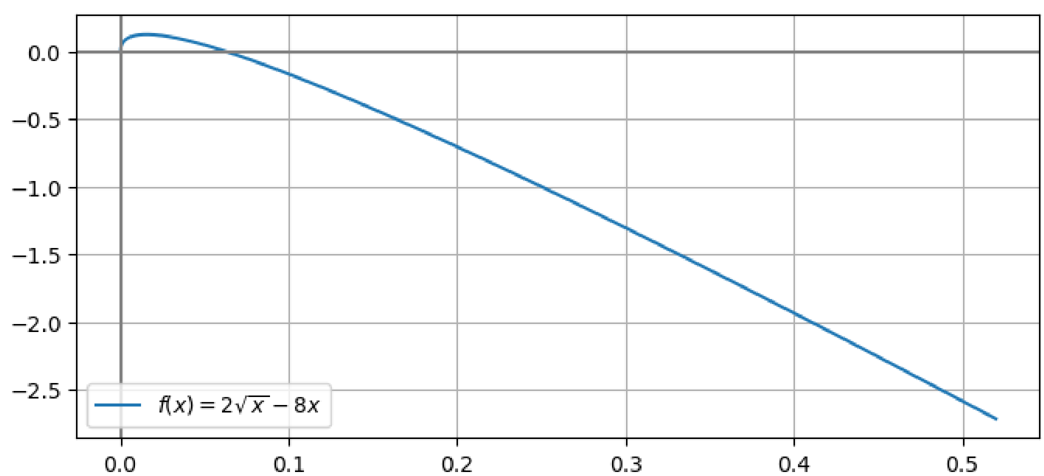


Figure 3: The graph of $f(x) = 2\sqrt{x} - 8x$

7. (1 point) Suppose that

$$f(x) = (x^2 + 12)(1 - x^2).$$

(A) Find all critical values of f. If there are no critical values, enter None. If there are more than one, enter them separated by commas.
Critical value(s) = ______

(B) Use interval notation to indicate where $f(x)$ is increasing.

Increasing:

____

(C) Use interval notation to indicate where $f(x)$ is decreasing.
Decreasing:

____

(D) Find the x-coordinates of all local maxima of f. If there are no local maxima, enter None. If there are more than one, enter them separated by commas.

Local maxima at $x =$ ______

(E) Find the x-coordinates of all local minima of f. If there are no local minima, enter None. If there are more than one, enter them separated by commas.

Local minima at $x =$ ______

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down:

____

Solution: (A) Notice that $f(x)$ is differentiable on the domain.
We first compute the derivative of f:

$$f'(x) = 2x(1 - x^2) + (x^2 + 12)(-2x) = -4x^3 - 22x = -2x(2x^2 + 11)$$

Then the equation $f'(x) = 0$ has only one solution $x = 0$. So there is only one critical point $x = 0$, and its value is $f(0) = 12$.

(B) The derivative $f'(x) = -2x(2x^2 + 11) > 0$ on $(-\infty, 0)$. As the critical point $x = 0$ is at the boundary of this interval, f is increasing on $(-\infty, 0]$.

(C) The derivative $f'(x) = -2x(2x^2 + 11) < 0$ on $(0, \infty)$. As the critical point $x = 0$ is at the boundary of this interval, f is decreasing on $[0, -\infty)$.

(D) Since f is differentiable on the domain and has only one critical point (at $x = 0$), it suffices to check only this point.
We compute the second derivative

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(-4x^3 - 22x) = -12x^2 - 22$$

As $f''(0) = -22 < 0$, $x = 0$ is a local maxima.
So the only local maxima of f is at $x = 0$.

(E) As shown in (D), f has no local minima.

(G) Since $f''(x) = -12x^2 - 22 < 0$ on the whole domain $(-\infty, \infty)$, f is concave down on $(-\infty, \infty)$.

8. (1 point) Suppose that

$$f(x) = x^{1/3}(x+3)^{2/3}$$

(A) Find all critical values of f. If there are no critical values, enter **None** . If there are more than one, enter them separated by commas.
Critical value(s) = ______

(B) Use interval notation to indicate where $f(x)$ is increasing.

Note: When using interval notation in WeBWorK, you use **I** for ∞, **-I** for $-\infty$, and **U** for the union symbol. If there are no values that satisfy the required condition, then enter "" without the quotation marks.

Increasing:

____

(C) Use interval notation to indicate where $f(x)$ is decreasing.
Decreasing:

____

(D) Find the x-coordinates of all local maxima of f. If there are no local maxima, enter **None** . If there are more than one, enter them separated by commas.

Local maxima at x = _____

(E) Find the x-coordinates of all local minima of f. If there are no local minima, enter **None** . If there are more than one, enter them separated by commas.

Local minima at x = _____

(F) Use interval notation to indicate where $f(x)$ is concave up.
Concave up:

____

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down:

____

Solution: **(A)** First, note that f is continuous on the domain abd not differentiable at $x = 0$ and $x = -3$. So $x = 0$ and $x = -3$ are two critical points.
We then compute the derivative of f:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} x^{1/3}(x+3)^{2/3} = \frac{1}{3}x^{-2/3} \cdot (x+3)^{2/3} + x^{1/3} \cdot \frac{2}{3}(x+3)^{-1/3} = \frac{x+1}{\sqrt[3]{x^2(x+3)}}$$

As the only solution for $f'(x) = \frac{x+1}{\sqrt[3]{x^2(x+3)}} = 0$ is $x = -1$, there is a critical point at $x = -1$.
So f has 3 critical points which are at $x = -3$, $x = -1$ and $x = 0$ respectively.

(B) The derivative $f'(x) = \dfrac{x+1}{\sqrt[3]{x^2(x+3)}} > 0$ if and only if $x \in (-\infty, -3) \cup (-1, 0) \cup (0, \infty)$. Since f is continuous at $x = -3$, $x = -1$ and $x = 0$, f is increasing on $(-\infty, -3]$ and on $[-1, \infty)$ respectively.

(C) The derivative $f'(x) = \dfrac{x+1}{\sqrt[3]{x^2(x+3)}} < 0$ if and only if $-3 < x < -1$. Since f is continuous at $x = -3$ and $x = -1$, f is decreasing on $[-3, -1]$.

(D) It suffices to check all the critical points.

Near $x = -1$, f is differentiable and $f'(x) < 0$ when $x < -1$ and $f'(x) > 0$ when $x > -1$. So $x = -1$ is a local minimum.
Near $x = -3$, f is differentiable and $f'(x) > 0$ when $x < -1$ and $f'(x) < 0$ when $x > -1$. So $x = -3$ is a local maximum.
Near $x = 0$, f is differentiable and $f'(x) > 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$. So $x = 0$ is not a local extremum.

So the local maximum of f is at $x = -3$.

(E) As shown in (D), the local minimum of f is at $x = -1$.

(F) We compute the second derivative of $f(x)$:

$$f''(x) = \frac{1}{\sqrt[3]{x^2(x+3)}} + (x+1)\frac{-1}{3}\frac{1}{\left(\sqrt[3]{x^2(x+3)}\right)^4} \cdot 3x(x+2) = -\frac{2x}{\left(x^2(x+3)\right)^{4/3}}$$

Since $\left(x^2(x+3)\right)^{4/3} > 0$ on the domain, we can see that $f''(x) > 0$ when $x \in (-\infty, -3) \cup (-3, 0)$ and $f''(x) < 0$ when $x > 0$.

So $f(x)$ is concave up on $(-\infty, -3)$ and on $(-3, 0)$ respectively.

(G) As shown in (F), f is concave down on $(0, \infty)$.

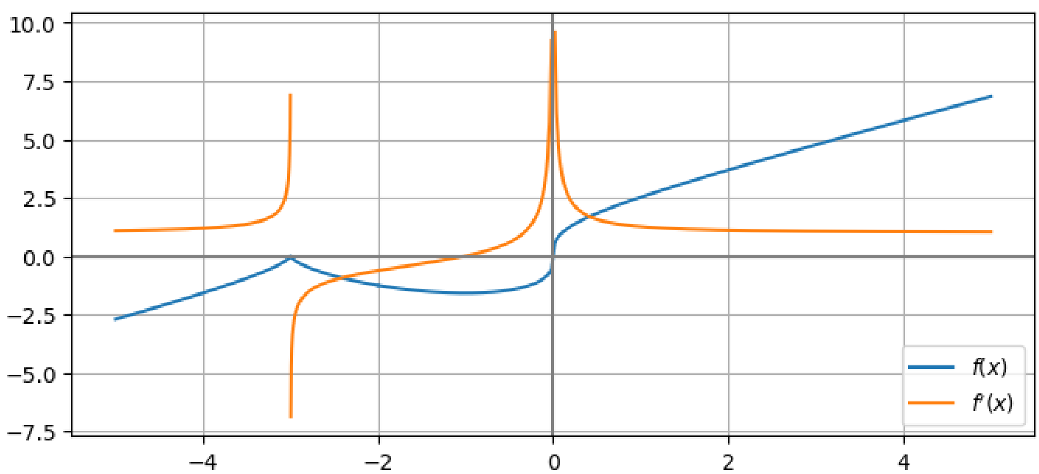
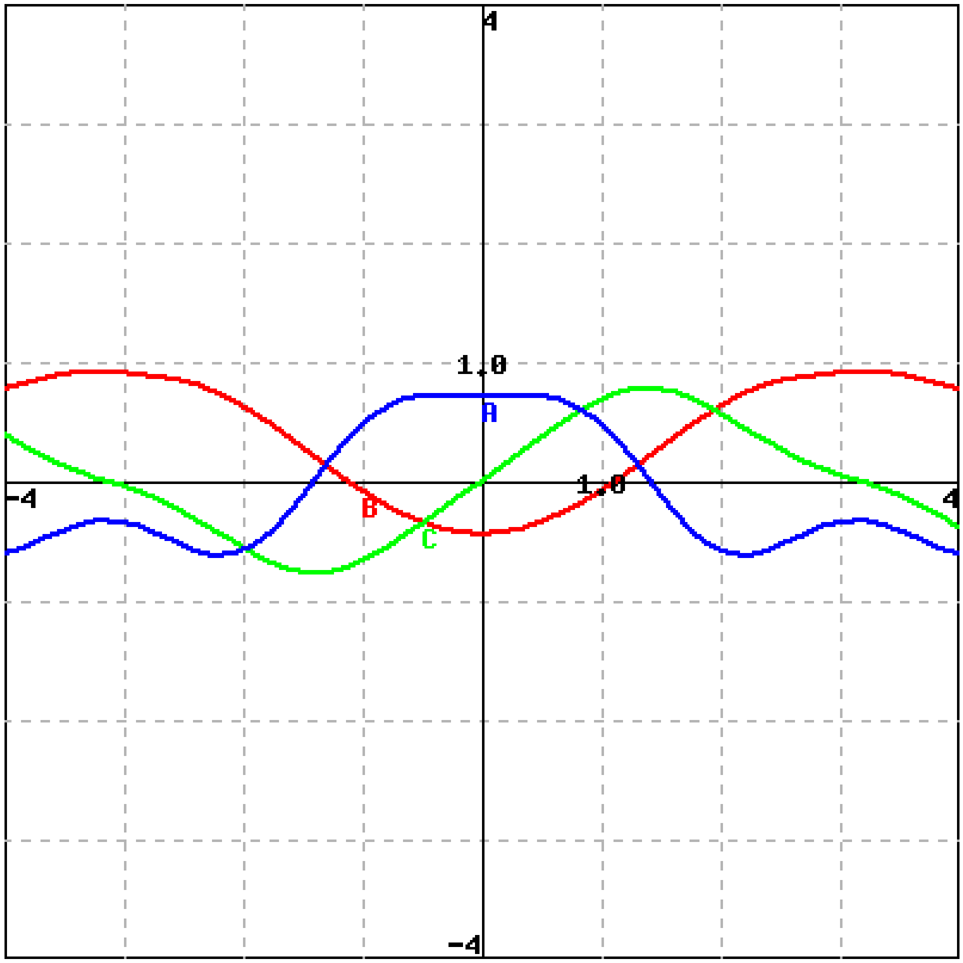


Figure 4: The graph of $f(x) = x^{1/3}(x+3)^{2/3}$ and its derivative

9. (1 point) 

Identify the graphs A (blue), B(red) and C (green) as the graphs of a function and its derivatives:

___ is the graph of the function

___ is the graph of the function's first derivative

___ is the graph of the function's second derivative

Solution: Call the functions in blue, red, and green f_B, f_R, f_G respectively.

At $x = 0$, only f_G takes the value 0, and the graphs of both f_B and f_R are flat near $x = 0$. So either $f'_B = f_G$ or $f'_R = f_G$.
On $(0, 1)$, f_B is nonincreasing but f_R is increasing. Since $f_G > 0$ on this interval, we can see that $f'_R = f_G$.

So either $f'_G = f_B$ or $f'_B = f_R$. Near $x = 1$, f_B is decreasing, but f_R changes sign from negative to positive. So we cannot have $f'_B = f_R$, and so we must have $f'_G = f_B$.

So f_R is the original function, f_G is the first derivative, and f_B is the second derivative.

10. (1 point) Suppose that

$$f(x) = 8x^2 \ln(x), \quad x > 0.$$

(A) List all the critical values of $f(x)$. Note: If there are no critical values, enter 'NONE'.

____

(B) Use interval notation to indicate where $f(x)$ is increasing.
Note: Use 'INF' for ∞, '-INF' for $-\infty$, and use 'U' for the union symbol. If there is no interval, enter 'NONE'.
Increasing: ________________

(C) Use interval notation to indicate where $f(x)$ is decreasing.
Decreasing: ________________

(D) List the x values of all local maxima of $f(x)$. If there are no local maxima, enter 'NONE'.
x values of local maximums = __________

(E) List the x values of all local minima of $f(x)$. If there are no local minima, enter 'NONE'.
x values of local minimums = __________

(F) Use interval notation to indicate where $f(x)$ is concave up.
Concave up: ________________

(G) Use interval notation to indicate where $f(x)$ is concave down.
Concave down: ________________

Solution: **(A)** It is easy to see that $f(x)$ is differentiable on the domain $(0, \infty)$. We compute its derivative

$$f'(x) = 16x \ln(x) + 8x^2 \cdot \frac{1}{x} = 16x \ln(x) + 8x = 8x(1 + 2\ln(x))$$

The equation $f'(x) = 8x(1 + 2\ln(x)) = 0$ has only one solution $x = e^{-\frac{1}{2}}$. So $f(x)$ has only one critical point, which is at $x = e^{-\frac{1}{2}}$.

(B) On the domain $(0, \infty)$, $8x > 0$. So $f'(x) > 0$ if and only if $1 + 2\ln(x) > 0$, which is on $(e^{-\frac{1}{2}}, \infty)$. Since $f(x)$ is continuous at $x = e^{-\frac{1}{2}}$, $f(x)$ is increasing on $[e^{-\frac{1}{2}}, \infty)$.

(C) Similar to the last question, $f(x) < 0$ when $1 + 2\ln(x) < 0$, which is $(0, e^{-\frac{1}{2}})$. Also, $f(x)$ is continuous at $x = e^{-\frac{1}{2}}$, so $f(x)$ is decreasing on $(0, e^{-\frac{1}{2}}]$.

(D) It suffices to check only the critical point. As the function is decreasing on the left of the critical point and increasing on the right, $x = e^{-\frac{1}{2}}$ is a local minima.

(E) As shown in the last question, $f(x)$ has no local maxima.

(F) We compute the second derivative

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(16x \ln(x) + 8x) = 16\left(\ln(x) + x \cdot \frac{1}{x}\right) + 8 = 8(2\ln(x) + 3)$$

Since $f''(x) > 0$ only when $x > e^{-\frac{3}{2}}$, $f(x)$ is concave up on $(e^{-\frac{3}{2}}, \infty)$.

(G) Similar to the last question, $f''(x) < 0$ only when $0 < x < e^{-\frac{3}{2}}$, so $f(x)$ is concave down on $(0, e^{-\frac{3}{2}})$.

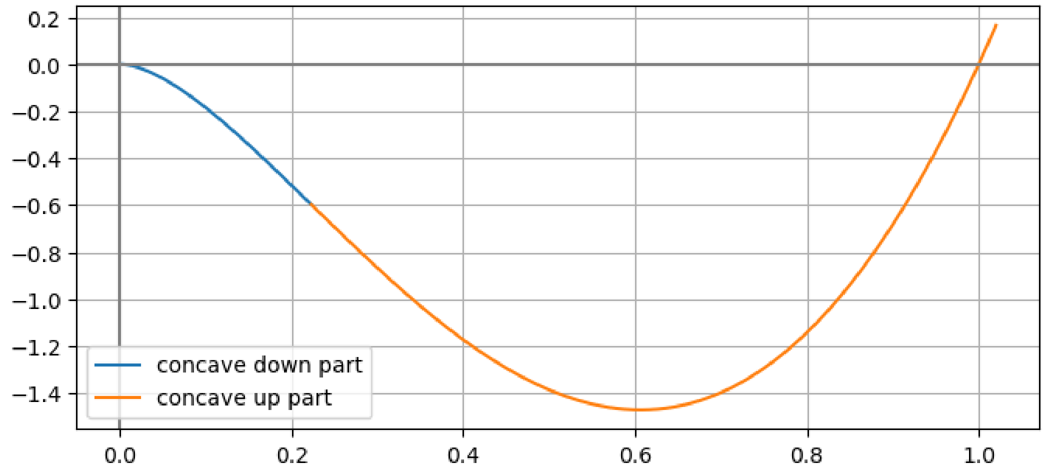


Figure 5: The grapg of $f(x) = 8x^2 \ln(x)$

11. (1 point) Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit $\lim_{x\to\infty}(\frac{10x^3+6x^2}{11x^3-4}) =$ __________
If the answer equals ∞ or $-\infty$, write INF or -INF in the blank.

Solution:

$$\begin{aligned}\lim_{x\to\infty}\frac{10x^3+6x^2}{11x^3-4} &= \lim_{x\to\infty}\frac{(10x^3+6x^2)'}{(11x^3-4)'} = \lim_{x\to\infty}\frac{10x^2+4x}{11x^2}\\ &= \lim_{x\to\infty}\frac{(10x^2+4x)'}{(11x^2)'} = \lim_{x\to\infty}\frac{10x+2}{11x}\\ &= \lim_{x\to\infty}\frac{(10x+2)'}{(11x)'} = \lim_{x\to\infty}\frac{10}{11}\\ &= \frac{10}{11}\end{aligned}$$

12. (1 point) Compute

$\lim_{x \longrightarrow 0} \dfrac{e^x - e^{-x}}{2\sin x} =$ ________________

Solution:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin x} = \lim_{x\to 0} \frac{(e^x - e^{-x})'}{(2\sin x)'} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2\cos x} = \frac{e^0 + e^0}{2\cos 0} = 1$$

13. (1 point) Let $f(x) = \dfrac{\ln x}{1 + (\ln x)^2}$ for x in $(0, \infty)$. Find

a) $\lim_{x \to 0^+} f(x) =$ __________

b) $\lim_{x \to \infty} f(x) =$ __________

Solution: a)

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\ln x}{1 + (\ln x)^2} = \lim_{x \to 0^+} \frac{(\ln x)'}{(1 + (\ln x)^2)'} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{2 \ln x \cdot \frac{1}{x}} = \lim_{x \to 0^+} \frac{1}{2 \ln x} = 0$$

b)

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln x}{1 + (\ln x)^2} = \lim_{x \to \infty} \frac{(\ln x)'}{(1 + (\ln x)^2)'} = \lim_{x \to \infty} \frac{\frac{1}{x}}{2 \ln x \cdot \frac{1}{x}} = \lim_{x \to \infty} \frac{1}{2 \ln x} = 0$$

14. (1 point) Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit $\lim_{t\to 0}(5\sin(5t)\ln(5t)) =$ ________
If the answer equals ∞ or $-\infty$, write INF or -INF in the blank.

Solution:

$$\begin{aligned}
\lim_{t\to 0}(5\sin(5t)\ln(5t)) &= \lim_{t\to 0}\frac{5\ln(5t)}{\frac{1}{\sin(5t)}} = \lim_{t\to 0}\frac{(5\ln(5t))'}{(\frac{1}{\sin(5t)})'} \\
&= \lim_{t\to 0}\frac{\frac{5}{t}}{-\frac{\cos(5t)}{\sin^2(5t)}} = \lim_{t\to 0} -\frac{5\sin^2(5t)}{t\cos(5t)} \\
&= -125\lim_{t\to 0}\frac{\sin^2(5t)}{(5t)^2}\cdot\lim_{t\to 0}\frac{t}{\cos(5t)} \\
&= 0
\end{aligned}$$

15. (1 point) Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit $\lim_{x\to 0}(\cot(14x) - \frac{1}{14x}) =$ __________
If the answer equals ∞ or $-\infty$, write INF or -INF in the blank.

Solution:

$$\begin{aligned}
\lim_{x\to 0}\left(\cot(14x) - \frac{1}{14x}\right) &= \lim_{14x\to 0}\left(\cot(14x) - \frac{1}{14x}\right) = \lim_{t\to 0}\left(\cot(t) - \frac{1}{t}\right) \\
&= \lim_{t\to 0}\frac{t\cos(t) - \sin(t)}{t\sin(t)} = \lim_{t\to 0}\frac{(t\cos(t) - \sin(t))'}{(t\sin(t))'} \\
&= \lim_{t\to 0}\frac{-t\sin(t)}{\sin(t) + t\cos(t)} = -\lim_{t\to 0}\frac{(t\sin(t))'}{(\sin(t) + t\cos(t))'} \\
&= \lim_{t\to 0}\frac{\sin(t) + t\cos(t)}{2\cos(t) - t\sin(t)} \\
&= \frac{\sin(0) + 0\cdot\cos(0)}{2\cos(0) - 0\cdot\sin(0)} \\
&= 0
\end{aligned}$$

16. (1 point) Evaluate the limit using L'Hôpital's rule if necessary

$$\lim_{x\to\infty}\left(1+\frac{8}{x}\right)^{\frac{x}{2}}$$

____

Solution: We first consider the limit $\lim_{x\to\infty}\frac{x}{2}\ln\left(1+\frac{8}{x}\right)$.

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{2}\ln\left(1+\frac{8}{x}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{8}{x}\right)}{\frac{2}{x}} = \lim_{x\to\infty}\frac{(\ln\left(1+\frac{8}{x}\right))'}{(\frac{2}{x})'} \\ &= \lim_{x\to\infty}\frac{\frac{x}{x+8}\cdot(-8x^{-2})}{-2x^{-2}} = 4\lim_{x\to\infty}\frac{x}{x+8} \\ &= 4\end{aligned}$$

So

$$\lim_{x\to\infty}\left(1+\frac{8}{x}\right)^{\frac{x}{2}} = \lim_{x\to\infty}e^{\frac{x}{2}\ln\left(1+\frac{8}{x}\right)} = e^{\lim_{x\to\infty}\frac{x}{2}\ln\left(1+\frac{8}{x}\right)} = e^4$$

Alternatively, this problem can be solved without using L'Hôpital's Rule:

Solution:

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{8}{x}\right)^{\frac{x}{2}} &= \lim_{x\to\infty}\left[\left(1+\frac{8}{x}\right)^{\frac{x}{8}}\right]^4 \\ &= \left[\lim_{\frac{x}{8}\to\infty}\left(1+\frac{8}{x}\right)^{\frac{x}{8}}\right]^4 = \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right]^4 \\ &= e^4\end{aligned}$$

where the well-known limit

$$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t = e$$

is applied.

17. (1 point) Apply L'Hôpital's Rule to evaluate the following limit. It may be necessary to apply it more than once.

$\lim_{x\to 0+} (\tan x)^{\sin x} =$ __________

Solution: We first consider the limit $\lim_{x\to 0+} \sin x \ln \tan x$

$$\begin{aligned}
\lim_{x\to 0+} \sin x \ln \tan x &= \lim_{x\to 0+} \frac{\ln \tan x}{\frac{1}{\sin x}} = \lim_{x\to 0+} \frac{(\ln \tan x)'}{(\frac{1}{\sin x})'} \\
&= \lim_{x\to 0+} \frac{\frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}}{-\frac{\cos x}{\sin^2 x}} = -\lim_{x\to 0+} \frac{\sin x}{\cos^2 x} \\
&= 0
\end{aligned}$$

So

$$\lim_{x\to 0+} (\tan x)^{\sin x} = \lim_{x\to 0+} e^{\sin x \ln \tan x} = e^{\lim_{x\to 0+} \sin x \ln \tan x} = e^0 = 1$$