THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Coursework 6

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(1) Find the critical point and determine if the function is increasing or decreasing on the given intervals.
$y = -x^2 + 8x + 9$
The critical point c is? And determine the monotonicity of y on

$$(-\infty, c)\ (c, \infty)$$

Solution:
The derivative of $y = -x^2 + 8x + 9$ is $y' = -2x + 8$.
This means $y' = 0$ when $x = 4$.
Therefore $c = 4$ is the critical point.
Note that

$$y' = -2x + 8 < 0 \text{ when } x > 4$$
$$y' = -2x + 8 > 0 \text{ when } x < 4$$

So y is increasing on $(-\infty, 4)$ and decreasing on $(4, \infty)$.

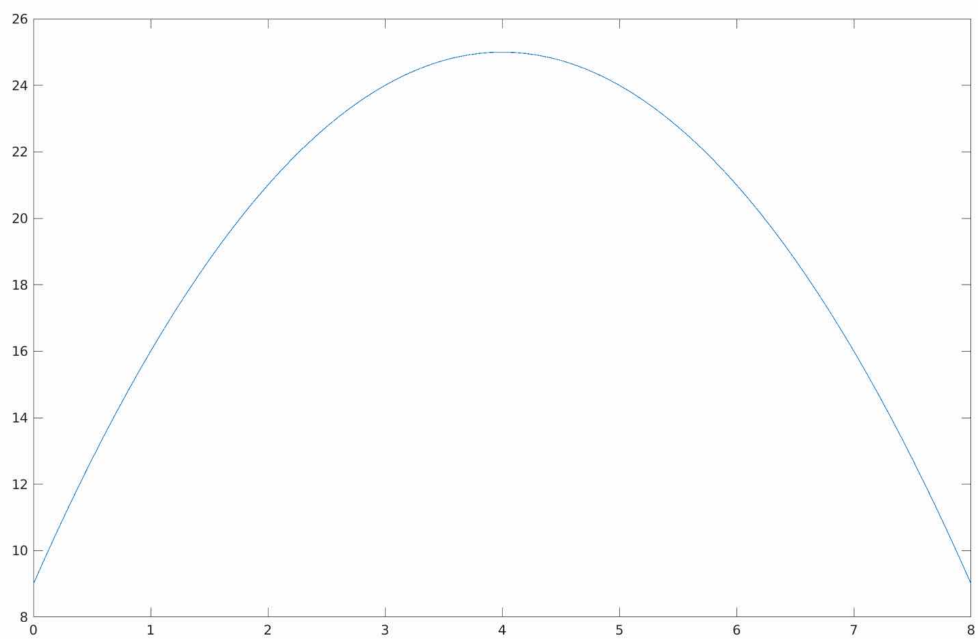


FIGURE 1. The graph of $f(x) = -x^2 + 8x + 9$

(2) Find the critical point and the interval on which the given function is increasing or decreasing, and apply the First Derivative Test to the critical point. Let

$$f(x) = 2x - 2\ln(8x)\,, x > 0$$

(a) Critical Point =?

(b) Is f a maximum or minimum at the critical point ?

(c) The **open** interval on the left of the critical point is ?
On this interval, f is ? (increasing/decreasing) while f' is ? (positive or negative)

(d) The **open** interval on the right of the critical point is ?
On this interval, f is ? (increasing/decreasing) while f' is ? (positive or negative)

Solution:

(a) Note that

$$f'(x) = 2 - \frac{2}{x}$$

So $f'(x) = 0$ when $x = 1$. So the critical point is $x = 1$.

(b) Let us compute the second derivative

$$f''(x) = \frac{2}{x^2}$$

Since $f''(x) > 0$ for all $x > 0$, it follows that $f'(x)$ is always increasing. So $f'(x)$ is negative on $(0,1)$ and positive on $(1,\infty)$. That means $f(x)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$. Hence $f(x)$ is minimum at the critical point $x = 1$.

(c) As shown in (b), the interval on the left of the critical point is $(0,1)$, $f(x)$ is decreasing and $f'(x)$ is negative on this interval.

(d) As shown in (b), the interval on the right of the critical point is $(1,\infty)$, $f(x)$ is increasing and $f'(x)$ is positive on this interval.

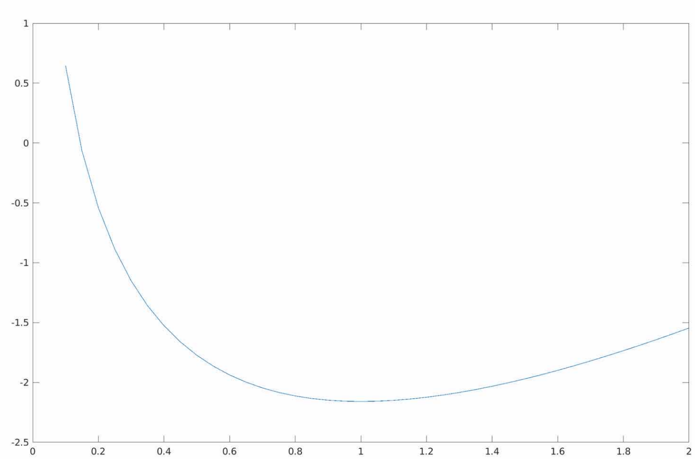
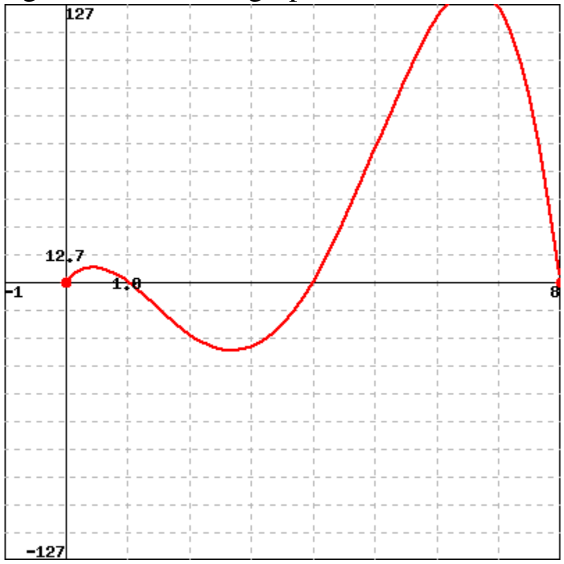


FIGURE 2. The graph of $f(x) = 2x - 2\ln(8x)$

(3) Determine the **open** intervals on which f is increasing or decreasing, assuming the figure below is the graph of the derivative of f.



Solution:

As shown in the graph, we can see that $f'(x)$ is negative on $(1,4)$ and positive on $(0,1)\cup(4,8)$. So according to the relations between functions and their derivatives, $f(x)$ is decreasing on $(1,4)$ and increasing on $(0,1)$, $(4,8)$

(4) Let a, h, and k be arbitrary real numbers with $a \neq 0$, and let f be the function given by the rule $f(x) = a(x-h)^2 + k$.
Which of the following statements are true about f? Select all that apply.

(a) The graph of $y = f(x)$ is a line.

(b) The graph of $y = f(x)$ is a parabola.

(c) An extreme value occurs at the point (h,k).

(d) If $a > 0$, then f has a global maximum.

(e) If $a > 0$, then f has a global minimum.

(f) The maximum value of $f(x)$ is h.

(g) None of the above
Next we use some calculus to develop familiar ideas from a different perspective.
To start, treat a, h, and k as constants and compute $f'(x)$.
$f'(x) =$ __________
Find a critical value of f. (This will depend on at least one of a, h, and k.)
Critical value = __________
Assume that $a < 0$. Make a derivative sign chart for f. Based on this information, classify the critical value of f as a maximum or minimum.

Solution:

(a) Since $f(x) = a(x-h)^2 + k$ with $a \neq 0$ has a quadratic term, the graph is not a line.
(b) As we can see in (a), the graph is a parabola.
(c) If $a > 0$, we have $a(x-h)^2 \geq 0$, so $f(x) = a(x-h)^2 + k \geq k = f(h)$.
On the other hand, if $a < 0$, we have $a(x-h)^2 \leq 0$, so $f(x) = a(x-h)^2 + k \leq k = f(h)$.
In both cases, $f(h) = k$ is an extreme value, and (h,k) is an extreme point.
(d) As shown in (c), $k = f(h)$ is not a maximum, so it is not a global maximum.
(e) As shown in (c), $f(x)$ has a global minimum $f(h) = k$ when $a > 0$.
(f) According to (d) and (e), it depends on whether a is positive or negative. So it is not true in general.
(g) Since (b), (c), and (e) are correct, this option is not correct.

The derivative of $f(x)$, treating a,h,k as constants, is $f'(x) = 2a(x-h)$.
So the critical point of $f(x)$ is c where $f'(c) = 2a(c-h) = 0$, which implies $c = h$ as $a \neq 0$. The critical value is then $f(c) = f(h) = k$.
When $a < 0$, we have the second derivative $f''(x) = 2a < 0$. So $f'(x) = 2a(x-h)$ is always strictly decreasing. As $f'(h) = 0$, we have

$$f'(x) \text{ is positive on } (-\infty, h)$$
$$f'(x) \text{ is negative on } (h, \infty)$$

The sign chart for f' is then

x	$x < h$	$x = h$	$x > h$
$f'(x)$	Positive	0	Negative

So

$$f(x) \text{ is increasing on } (-\infty, h)$$
$$f(x) \text{ is decreasing on } (h, \infty)$$

Hence the critical $f(h) = k$ is a maximum.
We can also plot the graph for $f(x)$ choosing some values of a, h, k:

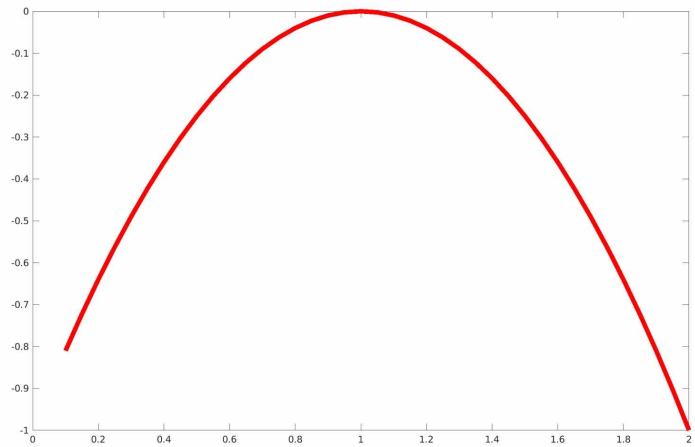


FIGURE 3. The graph of $f(x) = a(x-h)^2 + k$ for $a = -1$, $h = 1$, $k = 0$

If, on the other hand, $a > 0$, we would then have

x	$x < h$	$x = h$	$x > h$
$f'(x)$	Negative	0	Positive

So the critical value k would be a minimum.
The graph of $f(x)$ would look like the following one:

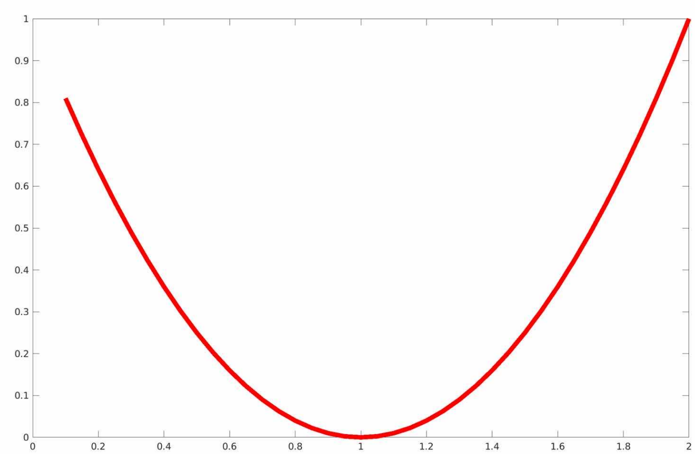


FIGURE 4. The graph of $f(x) = a(x-h)^2 + k$ for $a = 1$, $h = 1$, $k = 0$

(5) The function

$$f(x) = -6x^3 - 9x^2 + 216x + 4$$

is increasing on the interval [___ , ___].
It is decreasing on the interval ($-\infty$, ___] and the interval [___, ∞).

Solution:

We can compute the derivate of $f(x)$:

$$f'(x) = -18x^2 - 18x + 216$$

Let us find the critical point:

$$-18x^2 - 18x + 216 = -18(x-3)(x+4) = 0 \Longrightarrow x = -4 \text{ or } x = 3$$

By our discussion in (4),

x	$x < -4$	$-4 < x < 3$	$x > 3$
$f'(x)$	Negative	Positive	Negative
$f(x)$	Decreasing	Increasing	Decreasing

So $f(x)$ is increasing on the interval $[-4, 3]$ and decreasing on the intervals $(-\infty, -4]$ and $[3, \infty)$.

(6) For the function $f(x) = e^{4x} + e^{-x}$ defined on the interval $[-3, \infty)$, find all intervals where the function is increasing and decreasing.
f is increasing on ______________
f is decreasing on ______________
(Give your answer as an interval or a list of intervals, e.g., **(-inf,8]** *or* **(1,5),(7,10)** .
Solution: The derivative of $f(x)$ is

$$f'(x) = 4e^{4x} - e^{-x}$$

and the second derivative is

$$f''(x) = 16e^{4x} + e^{-x}$$

Notice that $f''(x)$ is always positive on the domain $[-3, \infty)$, so $f'(x)$ is always increasing on the domain. Solving $f'(x) = 4e^{4x} - e^{-x} = 0$, we have $x = -\frac{2}{5}\ln(2) > -3$. It follows that:

x	$-3 < x < -\frac{2}{5}\ln(2)$	$x > -\frac{2}{5}\ln(2)$
$f'(x)$	Negative	Positive
$f(x)$	Decreasing	Increasing

So f is increasing on $\left[-\frac{2}{5}\ln(2), \infty\right)$ and decreasing on $\left[-3, -\frac{2}{5}\ln(2)\right]$.

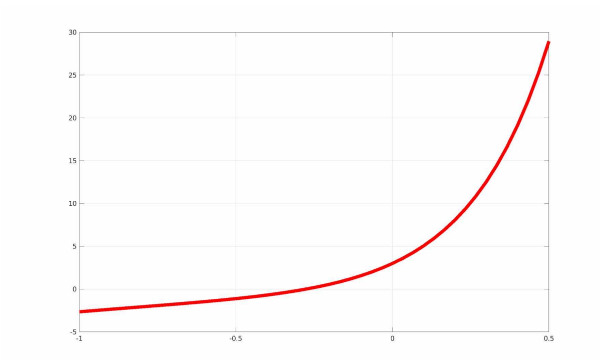


FIGURE 5. The graph of $f'(x)$

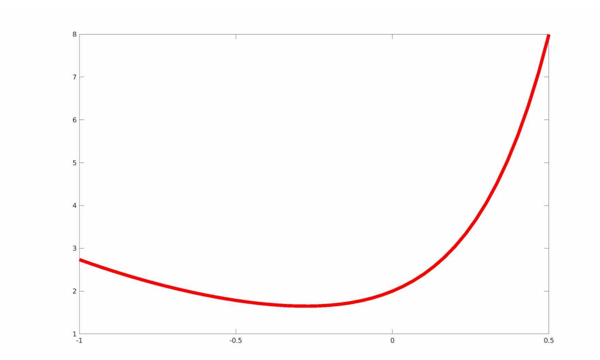


FIGURE 6. The graph of $f(x)$

(7)

$$f(x) = \frac{x}{x^2+11x+24}$$

a) Give the domain of f (in interval notation) ________

b) Determine the intervals on which f is increasing and decreasing.

f is increasing on: ________

f is decreasing on: ________

Solution:

(a) Since the denominator cannot be 0, the domain of $f(x)$ is where $x^2+11x+24 = (x+8)(x+3) \neq 0$. So the domain of $f(x)$ is

$$(-\infty,-8)\cup(-8,-3)\cup(-3,\infty)$$

(b) Note that

$$f'(x) = \frac{-x^2+24}{(x^2+11x+24)^2}$$

Solving $f'(x) = 0$, we have $x = -2\sqrt{6}$ or $x = 2\sqrt{6}$.

We should then have

x	$x < -2\sqrt{6}$	$-2\sqrt{6} < x < 2\sqrt{6}$	$x > 2\sqrt{6}$
$f'(x)$	Negative	Positive	Negative
$f(x)$	Decreasing	Increasing	Decreasing

However, $f(x)$ is not defined on $x = -3$ and $x = -8$, so we have to remove these two points from our result. Comparing the values, we have $-8 < -2\sqrt{6} < -3 < 2\sqrt{6}$. Therefore the correct chart is

x	$x < -8$	$-8 < x < -2\sqrt{6}$	$-2\sqrt{6} < x < -3$	$-3 < x < 2\sqrt{6}$	$x > 2\sqrt{6}$
$f'(x)$	Negative	Negative	Positive	Positive	Negative
$f(x)$	Decreasing	Decreasing	Increasing	Increasing	Decreasing

Hence $f(x)$ is increasing on

$$[-2\sqrt{6},-3),(-3,2\sqrt{6}]$$

and decreasing on

$$(-\infty,-8),(-8,-2\sqrt{6}],[2\sqrt{6},\infty)$$

(8) Let

$$f(x) = 6x + \frac{4}{x}.$$

(A) Use interval notation to indicate where $f(x)$ is increasing.

Increasing: ____________________

(B) Use interval notation to indicate where $f(x)$ is decreasing.

Decreasing: ____________________

Solution:

As usual, let us first compute the derivative:

$$f'(x) = 6 - \frac{4}{x^2}$$

Note that $f(x)$ is not defined on $x = 0$ as the denominator of $\frac{4}{x}$ would be 0, and $f'(x) = 0$ if $x = \frac{\sqrt{6}}{3}$ or $x = -\frac{\sqrt{6}}{3}$, Therefore we have the following chart:

x	$x < -\frac{\sqrt{6}}{3}$	$-\frac{\sqrt{6}}{3} < x < 0$	$0 < x < \frac{\sqrt{6}}{3}$	$x > \frac{\sqrt{6}}{3}$
$f'(x)$	Positive	Negative	Negative	Positive
$f(x)$	Increasing	Decreasing	Decreasing	Increasing

So f is increasing on

$$\left(-\infty, -\frac{\sqrt{6}}{3}\right], \left[\frac{\sqrt{6}}{3}, \infty\right)$$

and f is decreasing on

$$\left[-\frac{\sqrt{6}}{3}, 0\right), \left(0, \frac{\sqrt{6}}{3}\right]$$

(9) Let $f(x) = 8\sqrt{x} - 8x$ for $x > 0$. Find the open intervals on which f is increasing (decreasing).

1. f is increasing on the intervals ____________
2. f is decreasing on the intervals ____________

Notes: In the first two, your answer should either be a single interval, such as (0,1), a comma separated list of intervals, such as (-inf, 2), (3,4), or the word "none".

Solution:

Computing the derivative, we have

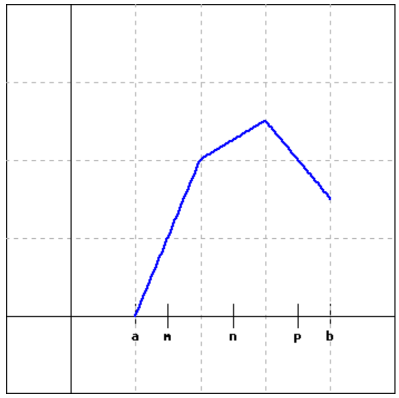
$$f'(x) = \frac{4}{\sqrt{x}} - 8$$

Note that $f'(x) = 0$ if $x = \frac{1}{4}$. So we have

x	$0 < x < \frac{1}{4}$	$x > \frac{1}{4}$
$f'(x)$	Positive	Negative
$f(x)$	Increasing	Decreasing

So $f(x)$ is increasing on $\left(0, \frac{1}{4}\right)$ and decreasing on $\left(\frac{1}{4}, \infty\right)$.

12 (10) Consider the function graphed below.



Does this function satisfy the hypotheses of the Mean Value Theorem on the interval $[a,b]$? [yes/no]

Does it satisfy the conclusion? [yes/no]

At what point c is $f'(c) = \dfrac{f(b) - f(a)}{b - a}$?

Solution: Recall that the Mean Value Theorem for the function $f(x)$ defined on $[a,b]$ requires that

(a) $f(x)$ is continuous on $[a,b]$

(b) $f(x)$ is differentiable on (a,b)

Although $f(x)$ is continuous on $[a,b]$, we can see that the function $f(x)$ is not differentiable at two points in the interval (a,b). Thus it does not satisfy the hypotheses of mean value theorem.

However, the conclusion may still be correct. For example, if the slope at point n is exactly $\dfrac{f(b) - f(a)}{b - a}$, the conclusion is then satisfied at point n.

(11) Consider the functions $f(x) = e^{x-1} - 1$ and $g(x) = x - 1$. These are continuous and differentiable for $x > 0$. In this problem we use the Racetrack Principle to show that one of these functions is greater than the other, except at one point where they are equal.

(a) Find a point c such that $f(c) = g(c)$. $c =$ _______

(b) Find the equation of the tangent line to $f(x) = e^{x-1} - 1$ at $x = c$ for the value of c that you found in (a).

$y =$ _________________

(c) Based on your work in (a) and (b), what can you say about the derivatives of f and g?

$f'(x)$?[</=/>] $g'(x)$ for $0 < x < c$, and

$f'(x)$?[</=/>] $g'(x)$ for $c < x < \infty$.

(d) Therefore, the Racetrack Principle gives

$f(x)$?[<=/=/>=] $g(x)$ for $x \leq c$, and

$f(x)$?[<=/=/>=] $g(x)$ for $x \geq c$.

Solution:

(a) Note that $f(1) = e^{1-1} - 1 = 0 = 1 - 1 = g(1)$, so $c = 1$.

(b) As $f'(x) = e^{x-1}$, we have $f'(1) = 1$, so the tangent line of $f(x)$ at $x = 1$ is $y = f'(1)(x-1) + f(1) = x - 1$ (which happens to be $y = g(x)$).

(c) From (b), we can see that
  - (i) For $x > 1$, $f'(x) = e^{x-1} > e^0 = 1 = g'(x)$
  - (ii) For $0 < x < 1$, $f'(x) = e^{x-1} < e^0 = 1 = g'(x)$

  So $f'(x) < g'(x)$ for $0 < x < 1$ and $f'(x) > g'(x)$ for $1 < x < \infty$.

(d) As $f(1) = g(1)$, by Racetrack Principle,
  - (i) $f(x) \geq g(x)$ for $x \leq 1$
  - (ii) $f(x) \geq g(x)$ for $x \geq 1$

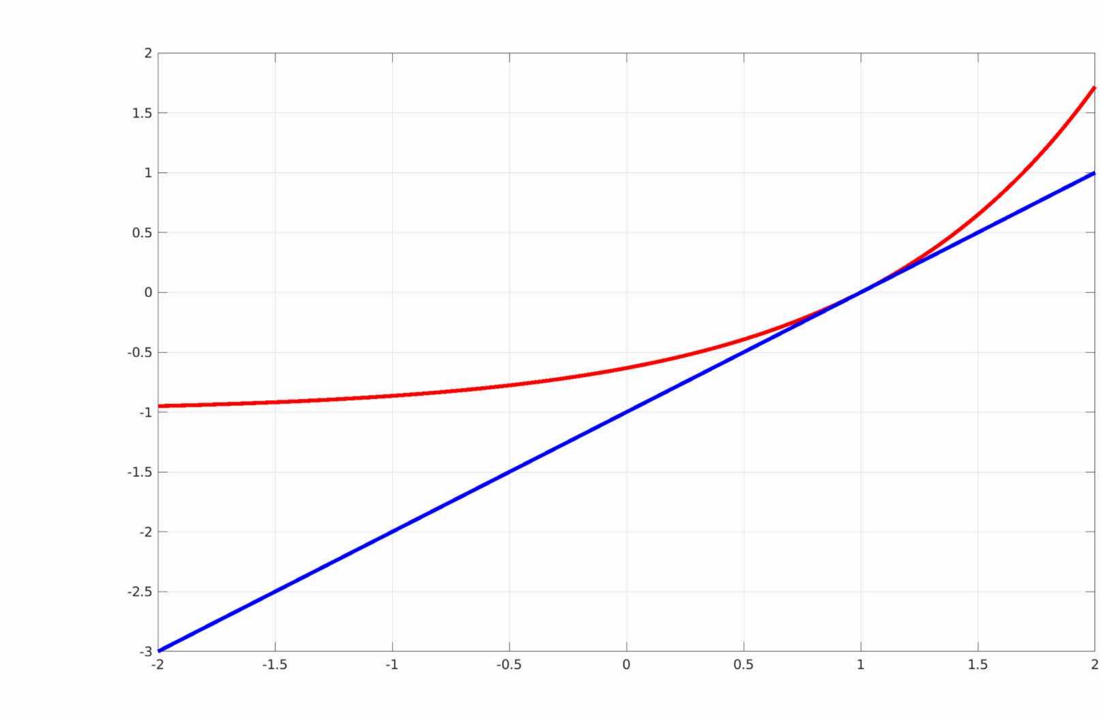


FIGURE 7. The graph of $f(x)$ (in red) and $g(x)$ (in blue)

(12) Use an appropriate theorem to complete the following statement.

If f is differentiable and $f(0) > f(3)$, then there is a number c, in the interval (___, ___) such that $f'(c)$?[</=/>] ___

What theorem guarantees this?

- The Mean Value Theorem
- The Increasing Function Theorem
- The Constant Function Theorem
- The Racetrack Principle

(Be sure that you can carefully apply this theorem to obtain the indicated result!)

Solution: Recall that

(a) The Increasing Function Theorem requires $f'(x) > 0$

(b) The Constant Function Theorem requires $f'(x) = 0$

(c) The Racetrack Principle requires two functions $f(x), g(x)$ and $f'(x) > g'(x)$

So the only theorem satisfying the condition of this problem is the Mean Value Theorem. According to the Mean Value Theorem, there exists $c \in (0,3)$, such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} < 0$$

(13) Suppose $f(x)$ is continuous on $[2,6]$ and $-4 \leq f'(x) \leq 5$ for all x in $(2,6)$. Use the Mean Value Theorem to estimate $f(6) - f(2)$.

Answer: ____ $\leq f(6) - f(2) \leq$ ____
Solution: According to The Mean Value theorem. There exists $c \in (2,6)$ such that:

$$f'(c) = \frac{f(6) - f(2)}{6 - 2} = \frac{1}{4}(f(6) - f(2))$$

Then we have:

$$-4 \leq f'(c) = \frac{1}{4}(f(6) - f(2)) \leq 5$$

Thus,

$$-16 \leq f(6) - f(2) \leq 20$$

The bounds can be achieved if on the domain $[2,6]$ the function $f(x)$ is a straight line of the corresponding slope.