THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Coursework 5

If you find any errors or typos, please email us at
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(1) Differentiate the following function:

$$f(t) = \sqrt[4]{t} - \frac{1}{\sqrt[4]{t}}$$

$f'(t) =$ __________.

Solution:

$$\begin{aligned} f'(t) &= (t^{\frac{1}{4}} - t^{-\frac{1}{4}})' \\ &= \frac{1}{4}t^{-\frac{3}{4}} - (-\frac{1}{4})t^{-\frac{5}{4}} \\ &= \frac{1}{4}(t^{-\frac{3}{4}} + t^{-\frac{5}{4}}). \end{aligned}$$

(2) Find the derivative of the function.

$$y = \sqrt{x}e^{(x^2)}(x^2+9)^{10}$$

$y' =$ ____________________

Solution:

$$\begin{aligned} y' &= (\sqrt{x})'(e^{x^2}(x^2+9)^{10}) + \sqrt{x}(e^{x^2}(x^2+9)^{10})' \\ &= (\sqrt{x})'(e^{x^2}(x^2+9)^{10}) + \sqrt{x}(e^{x^2})'(x^2+9)^{10} + \sqrt{x}(e^{x^2})\left[(x^2+9)^{10}\right]' \\ &= \frac{e^{x^2}(x^2+9)^{10}}{2\sqrt{x}} + \sqrt{x}e^{x^2}(2x)(x^2+9)^{10} + \sqrt{x}e^{x^2}10(x^2+9)^9(2x). \\ &= e^{x^2}\sqrt{x}\left[\frac{(x^2+9)^{10}}{2x} + 2x(x^2+9)^{10} + 20x(x^2+9)^9\right]. \end{aligned}$$

(3) If

$$f(x) = \frac{4\sin x}{1+\cos x}$$

find $f'(x)$.

Solution:

$$\begin{aligned} f'(x) &= \frac{(4\sin x)'(1+\cos x) - (4\sin x)(1+\cos x)'}{(1+\cos x)^2} \\ &= \frac{4\cos x(1+\cos x) - 4\sin x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{4\cos x + 4\cos^2 x + 4\sin^2 x}{(1+\cos x)^2} \\ &= \frac{4\cos x + 4}{(1+\cos x)^2} \\ &= \frac{4}{1+\cos x)}. \end{aligned}$$

(4) Calculate the derivative using the appropriate rule or combination of rules.

$$f(x) = \frac{e^x}{(e^x + 4)(x + 3)}$$

$f'(x) =$ ________________________

Solution: To compute $f'(x)$ we begin with quotient rule

$$f'(x) = \frac{(e^x + 4)(x + 3)\frac{d}{dx}[e^x] - e^x \frac{d}{dx}[(e^x + 4)(x + 3)]}{((e^x + 4)(x + 3))^2}.$$

Next, recall that $\frac{d}{dx}[e^x] = e^x$, and use the product rule to compute

$$\frac{d}{dx}[(e^x + 4)(x + 3)] = \frac{d}{dx}[e^x + 4](x + 3) + (e^x + 4)\frac{d}{dx}[x + 3]$$

which is

$$(e^x)(x + 3) + (e^x + 4)(1).$$

Therefore

$$f'(x) = \frac{(e^x + 4)(x + 3) \cdot e^x - e^x \cdot (e^x(x + 3) + (e^x + 4))}{((e^x + 4)(x + 3))^2}$$

and after factoring out e^x in the numerator, expanding $(e^x + 4)(x + 3) = xe^x + 3 \cdot e^x + 4x + 4 \cdot 3$, and distributing the minus sign, we get

$$f'(x) = \frac{e^x(xe^x + 4x + 3e^x - xe^x - 3e^x - e^x - 4)}{((e^x + 4)(x + 3))^2}$$

which simplifies to

$$f'(x) = \frac{e^x(4x - e^x - 4)}{((e^x + 4)(x + 3))^2}.$$

(5) Find the derivative of
$f(y) = e^{e^{(y^7)}}$

Solution: $f'(y) = 7y^6 e^{y^7} e^{e^{y^7}}$.

(6) Find $\frac{dy}{dx}$ when

$$y = -5\frac{1}{\sin^3(\ln(x))}$$

Solution: Differentiating gives

$$\frac{dy}{dx} = 5\frac{3\sin^2(\ln(x))\,\frac{1}{x}\cos(\ln(x))}{\left(\sin^3(\ln(x))\right)^2} = \frac{15\cos(\ln(x))}{x\sin^4(\ln(x))}.$$

(7) Differentiate $g(x) = \ln\left(\frac{5-x}{5+x}\right)$.

Solution:

$$\begin{aligned} g'(x) &= \frac{5+x}{5-x}\cdot\left(\frac{5-x}{5+x}\right)' \\ &= \frac{5+x}{5-x}\cdot\frac{(-1)(5+x)-(5-x)}{(5+x)^2} \\ &= \frac{-5-x-5+x}{(5+x)(5-x)} \\ &= \frac{10}{x^2-25}. \end{aligned}$$

(8) (a) Using laws of logarithms, write the expression below using sums and/or differences of logarithmic expressions which do not contain the logarithms of products, quotients, or powers.

$$\ln\sqrt{\frac{(x-9)^{38}}{(2x-7)^{26}}} = \underline{\hspace{4cm}}$$

Hint: $\sqrt{u^2} = |u|$.

(b) Use your answer from part (a) to evaluate the derivative.

$$\frac{d}{dx}\left(\ln\sqrt{\frac{(x-9)^{38}}{(2x-7)^{26}}}\right) = \underline{\hspace{4cm}}$$

Solution:

(a) $\ln\sqrt{\frac{(x-9)^{38}}{(2x-7)^{26}}} = \ln\frac{|x-9|^{19}}{|2x-7|^{13}} = 19\ln|x-9| - 13\ln|2x-7|.$

(b)

$$\begin{aligned}\left(\ln\sqrt{\frac{(x-9)^{38}}{(2x-7)^{26}}}\right) &= \frac{19}{|x-9|}\cdot\frac{d|x-9|}{dx} - \frac{13}{|2x-7|}\cdot\frac{d|2x-7|}{dx}\\ &= \frac{19}{|x-9|}\cdot\frac{x-9}{|x-9|} - \frac{13}{|2x-7|}\cdot\frac{2x-7}{|2x-7|}\cdot 2\\ &= \frac{19(x-9)}{(x-9)^2} - \frac{26(2x-7)}{(2x-7)^2}\\ &= \frac{19}{(x-9)} - \frac{26}{(2x-7)},\end{aligned}$$

because $\frac{d|u|}{dx} = \frac{d\sqrt{u^2}}{dx} = \frac{1}{2}\frac{2u}{|u|}\frac{du}{dx} = \frac{u}{|u|}\frac{du}{dx}$.

(9) Let

$$f(x) = \ln[x^3(x+7)^6(x^2+9)^6],$$

$f'(x) = \underline{\hspace{4cm}}$.

Solution: First, use the properties of logarithms to express $f(x)$ as: $f(x) = 3\ln(x) + 6\ln(x+7) + 6\ln(x^2+9)$.

Therefore, $f'(x) = \frac{3}{x} + \frac{6}{x+7} + \frac{12x}{x^2+9}$.

(10) Find $\frac{dr}{dx}$ if

$$r = \frac{\ln 4x}{x^2\ln x^2} + \left(\ln\left(\frac{2}{x}\right)\right)^3$$

$\frac{dr}{dx} = \underline{\hspace{8cm}}$

Solution: Applying the quotient rule to the first summand involves an application of the product rule. Then apply the power and chain rules to the second summand. Your answer should be equivalent to the expression

$$\frac{1}{x^3\ln(x^2)} - \frac{2\ln(4x)}{x^3\ln(x^2)} - \frac{2\ln(4x)}{x^3\ln^2(x^2)} - \frac{3\ln^2\left(\frac{2}{x}\right)}{x}.$$

(11) Use the definition of a derivative to find $f'(x)$ and $f'(0)$ where:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Find the derivative of $f(x)$ for x not equal 0.
$f'(x) =$ ____________
(b) If the derivative does not exist enter DNE.
$f'(0) =$ ____________

Solution:
(a) $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$.
(b) Using the definition of the derivative we find that:

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$
$$= \lim_{h\to 0} h^2 \sin\left(\frac{1}{h}\right)\frac{1}{h}$$
$$= \lim_{h\to 0} h\sin(\frac{1}{h})$$
$= 0$, by the squeeze theorem.

(12) Let $f(x) = |x|\ln(7 - x)$. Find $f'(x)$.
Solution:
$x < 0$,

$$f(x) = -x\ln(7 - x),$$
$$f'(x) = -\ln(7 - x) + \frac{x}{7 - x}.$$

$0 < x < 7$,

$$f(x) = x\ln(7 - x),$$
$$f'(x) = \ln(7 - x) - \frac{x}{7 - x}.$$

$x = 0$,

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h\ln(7 - h) - 0}{h} = \lim_{h\to 0^+} \ln(7 - h) = \ln 7.$$
$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h\ln(7 - h) - 0}{h} = \lim_{h\to 0^+} -\ln(7 - h) = -\ln 7.$$

Since the limit of $\frac{f(h)-f(0)}{h}$ as $h \to 0$ doesn't exist, the derivative doesn't exist at $x = 0$.

(13) Compute $f'(x)$, $f''(x)$, $f'''(x)$, and then state a formula for $f^{(n)}(x)$, when

$$f(x) = -\frac{3}{x}$$

$f'(x) =$ ______________
$f''(x) =$ ______________
$f'''(x) =$ ______________
$f^{(n)}(x) =$ ______________

Solution: $f'(x) = \frac{d}{dx}\left[-\frac{3}{x}\right] = \frac{(-1)(-3)}{x^2} = \frac{3}{x^2}$
$f''(x) = \frac{d}{dx}\left[\frac{3}{x^2}\right] = \frac{(-2)(-1)(-3)}{x^3} = -\frac{6}{x^3}$

$f'''(x) = \frac{d}{dx}\left[-\frac{6}{x^3}\right] = \frac{(-3)(-2)(-1)(-3)}{x^4} = \frac{18}{x^4}$

Observing the pattern we get that, $f^{(n)}(x) = \frac{3(-1)^{n+1}(n!)}{x^{n+1}}$

(14) Find the 900th derivative of $f(x) = xe^{-x}$.

Answer: $f^{(900)}(x) =$ ______________

Solution: $f'(x) = (xe^{-x})' = e^{-x} + xe^{-x}(-1) = (1-x)e^{-x}$.

$f''(x) = ((1-x)e^{-x})' = (-1)e^{-x} + (1-x)e^{-x}(-1) = (x-2)e^{-x}$.

$f'''(x) = ((x-2)e^{-x})' = e^{-x} + (x-2)e^{-x}(-1) = (3-x)e^{-x}$.

$f^{(4)}(x) = ((3-x)e^{-x})' = (-1)e^{-x} + (3-x)e^{-x}(-1) = (x-4)e^{-x}$.

Observing the pattern we get that, $f^{(n)}(x) = (-1)^n(x-n)e^{-x}$.

Then, $f^{(900)}(x) = (x-900)e^{-x}$.

(15) Find $\frac{dy}{dx}$ if

$$5x^3y^2 - 4x^2y = 4.$$

Express your answer in terms of x, y if necessary.

$\dfrac{dy}{dx} =$ ______________

Solution: Taking the derivative with respect to x we get

$$0 = 15x^2y^2 + 10x^3y\frac{dy}{dx} - 8xy - 4x^2\frac{dy}{dx},$$

or

$$8xy - 15x^2y^2 = \left(10x^3y - 4x^2\right)\frac{dy}{dx}.$$

Therefore,

$$\frac{dx}{dy} = \frac{8xy - 15x^2y^2}{10x^3y - 4x^2}.$$

(16) Find $\dfrac{dy}{dx}$, if $y = \ln(3x^2 + y^2)$.

$\dfrac{dy}{dx} =$ __________

Solution: Writing the given equation as $e^y = 3x^2 + y^2$, and then differentiating implicitly with respect to x, gives

$$e^y\frac{dy}{dx} = 6x + 2y\frac{dy}{dx},$$

or

$$(e^y - 2y)\frac{dy}{dx} = 6x.$$

Therefore,

$$\frac{dy}{dx} = \frac{6x}{e^y - 2y}.$$

Note: Were the equation not revised before differentiating, the answer

$$\frac{dy}{dx} = \frac{6x}{3x^2 + y^2 - 2y}$$

would result.

(17) A parabola is defined by the equation

$$x^2 - 2xy + y^2 - 4x + 12 = 0$$

The parabola has horizontal tangent lines at the point(s) ______.

The parabola has vertical tangent lines at the point(s) ______.

Solution: Differentiating implicitly with respect to x gives

$$2x - 2y - 2x\,\frac{dy}{dx} + 2y\,\frac{dy}{dx} - 4 = 0,$$

or

$$(y - x)\,\frac{dy}{dx} = y - x + 2,$$

and so

$$\frac{dy}{dx} = \frac{y - x + 2}{y - x}.$$

The tangent line to the parabola is horizontal where $\frac{dy}{dx} = 0$, i.e., where $x - y = 2$. The equation of the parabola can be written in the form

$$(x - y)^2 - 4(x - y) + 12 - 4y = 0,$$

and $x - y = 2$ gives $8 = 4y$, or $y = 2$, and $x = 4$. Hence, the tangent line to the parabola is horizontal at the point $(4, 2)$ and nowhere else.

The tangent line the the parabola is vertical where

$$0 = \frac{dx}{dy} = \frac{y - x}{y - x + 2},$$

i.e., where $x - y = 0$. Together with the last displayed equation of the parabola, this gives $12 - 4y = 0$, or $y = 3$, and $x = 3$. Hence, the tangent line to the parabola is vertical at the point $(3, 3)$ and nowhere else.

(18) Let $y = (x^3 - 4x)^{\ln(x)}$. Find dy/dx .

Solution: Take logarithm, then differentiating both sides with respect to x:

$$\begin{aligned}
\frac{d}{dx}(\ln y) &= \frac{d}{dx}(\ln\left(x^3 - 4x\right)^{\ln(x)}) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left[\ln(x) \cdot \ln\left(x^3 - 4x\right)\right] \\
&= \frac{\ln\left(x^3 - 4x\right)}{x} + \frac{(\ln x)(3x^2 - 4)}{x^3 - 4x}. \\
\frac{dy}{dx} &= \left(x^3 - 4x\right)^{\ln(x)}\left(\frac{\ln\left(x^3 - 4x\right)}{x} + \frac{(\ln x)(3x^2 - 4)}{x^3 - 4x}\right).
\end{aligned}$$