THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Suggested Solution of Coursework 4

(1) Suppose

$$-5x - 26 \leq f(x) \leq x^2 + 3x - 10$$

Use this to compute the following limit.

$$\lim_{x \to -4} f(x)$$

What theorem did you use to arrive at your answer?

Solution:

Note that both sides tend to -6 as x tends to -4. Then, by the Squeeze Theorem,

$$\lim_{x \to -4} f(x) = -6$$

.

(2) Use the Squeeze Theorem to evaluate the following limit. If the answer is positive infinite, type "I"; if negative infinite, type "N"; and if it does not exist, type "D".

$$\lim_{x \to \infty} \frac{\sin x}{x}$$

[By using a graphing calculator, one can see that $f(x) = \dfrac{\sin x}{x}$ crosses its horizontal asymptote infinitely many times.]

Solution:

Note that for $x \geq 0$,

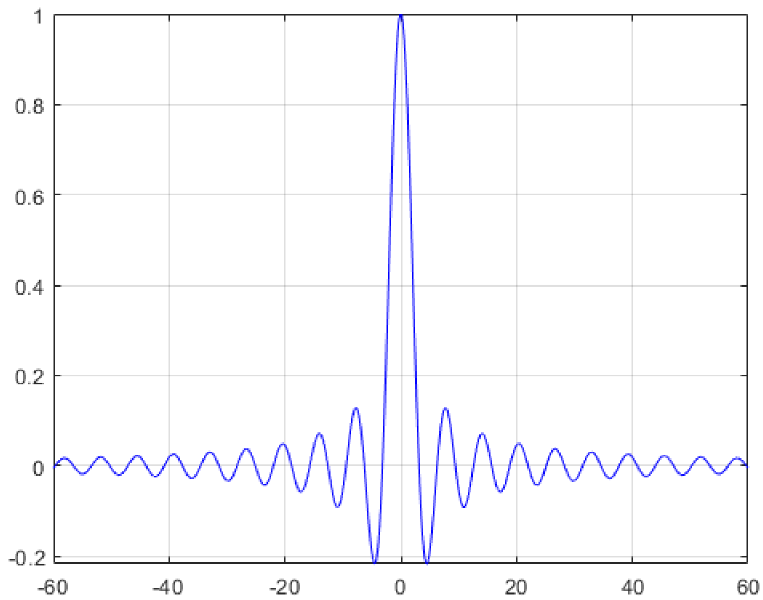
$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

.

By the Squeeze Theorem,

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

.

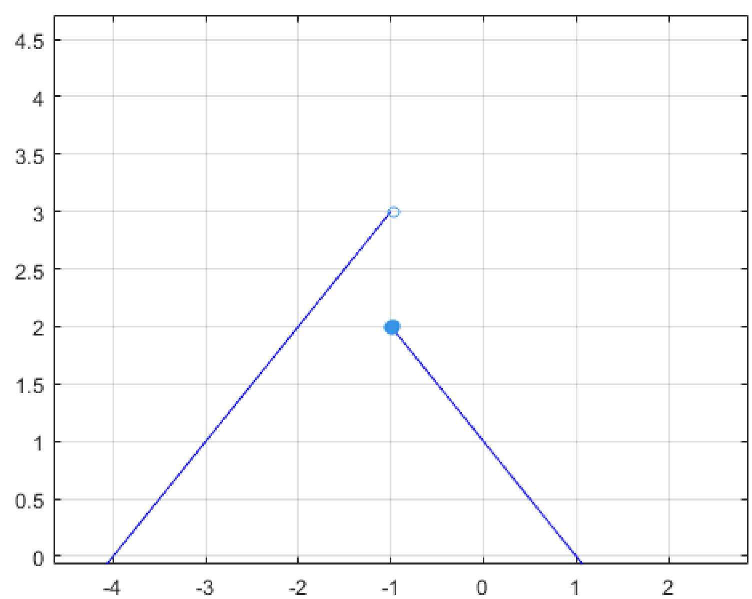


(3) Let

$$f(x) = \begin{cases} 4 + x, & x < -1, \\ 1 - x, & x \geq -1. \end{cases}$$

Find the indicated one-sided limits of f, and determine the continuity of f at the indicated point. You should also sketch a graph of $y = f(x)$, including hollow and solid circles in the appropriate places.

NOTE: Type DNE if a limit does not exist.



Solution:

$$\lim_{x \to -1^-} f(x) = 3$$
$$\lim_{x \to -1^+} f(x) = 2$$
$$\lim_{x \to -1} f(x) = \text{DNE}$$
$$f(-1) = 2$$

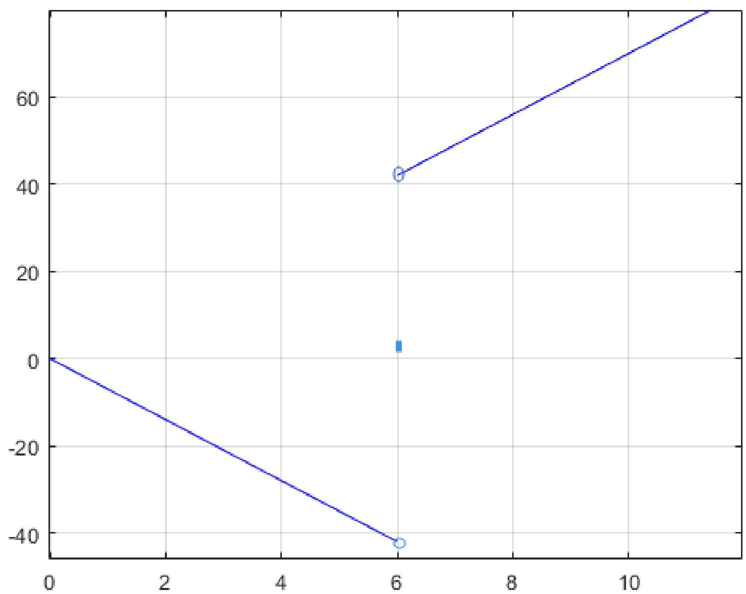
Is f continuous at -1?
No, since the limit of f at -1 does not exist.

(4) Let

$$f(x) = \begin{cases} -7x, & x < 6, \\ 1, & x = 6, \\ 7x, & x > 6. \end{cases}$$

Find the indicated one-sided limits of f, and determine the continuity of f at the indicated point. You should also sketch a graph of $y = f(x)$, including hollow and solid circles in the appropriate places.

NOTE: Type DNE if a limit does not exist.



Solution:

$\lim_{x\to 6^-} f(x) =$ -42
$\lim_{x\to 6^+} f(x) = 42$
$\lim_{x\to 6} f(x) =$ DNE
$f(6) = 1$

Is f continuous at $x = 6$?
No, since the limit of f at 6 does not exist.

(5) For what value of the constant c is the function f continuous on the interval $(-\infty, \infty)$?

$$f(x) = \begin{cases} x^2 - 10, & x \le c \\ 8x - 26, & x > c \end{cases}$$

$c =$ __________

Solution:

Suppose f be continuous at c. Then

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c} f(x)$$

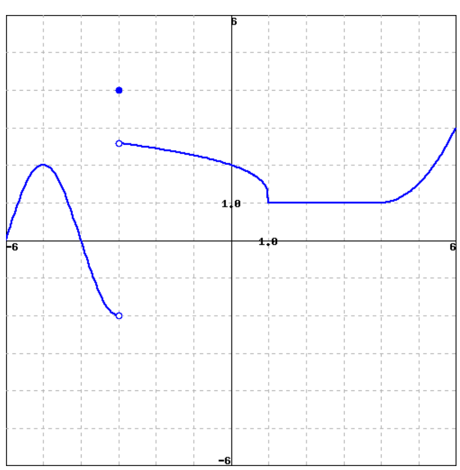
By solving the equation,

$$c^2 - 10 = 8c - 26$$

we obtain $c = 4$.

(6) Use the given graph of the function to find the x-values for which f is not differentiable.)

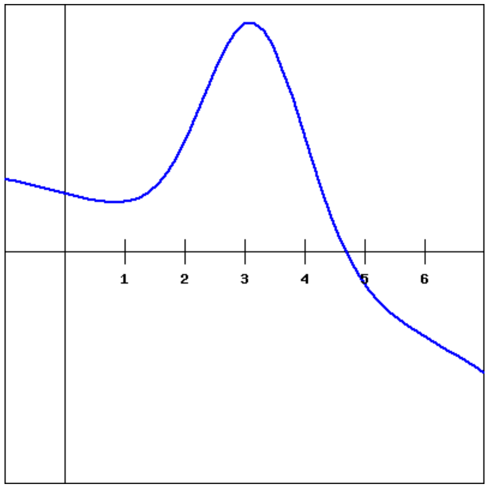
Answer (separate by commas): $x =$ ____________________



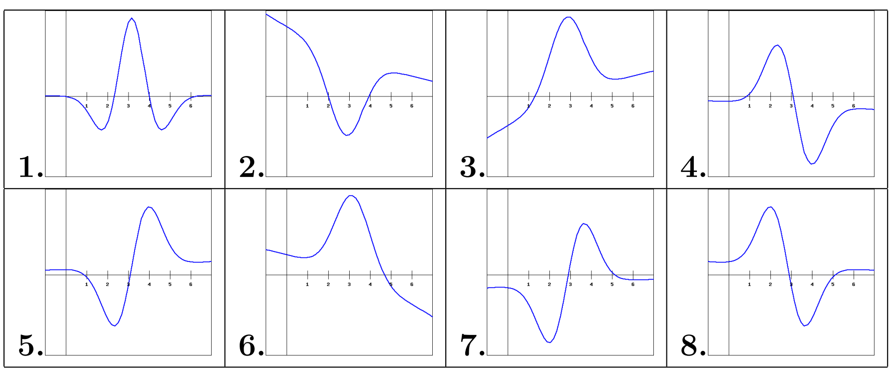
Solution:

$x = -3, 1.$

(7) For the function $f(x)$ shown in the graph below, sketch a graph of the derivative. You will then be picking which of the following is the correct derivative graph, but should be sure to first sketch the derivative yourself.



Which of the following graphs is the derivative of $f(x)$?



Solution:

Because the derivative gives the slope of the original function at each point x, we know that the derivative is negative where $f(x)$ is decreasing and positive where it is increasing. We can see the answer is 4 by observing two facts. The first is that f takes its maximum at $x = 3$. Hence the derivative vanishes at $x = 3$ and decreases around that point. The second is that f is decreasing for $5 \leq x \leq 6$. Hence the derivative is negative in this interval.

(8) Let $f(x) = |x - 7|$. Evaluate the following limits.

$$\lim_{x\to 7^-} \frac{f(x)-f(7)}{x-7} = \_\_\_\_\_$$

$$\lim_{x\to 7^+} \frac{f(x)-f(7)}{x-7} = \_\_\_\_\_$$

Thus the function $f(x)$ is not differentiable at 7.

Solution:

The first limit is -1 and the second limit is 1.

(9) Let

$$f(x) = \begin{cases} -(6x+3) & \text{if } x < -1 \\ 3x^2 & \text{if } -1 \le x \le 1 \\ 3x & \text{if } x > 1 \end{cases}$$

Find a formula for $f'(x)$.

Note: $f(x)$ is differentiable at $x = -1$, but is NOT differentiable at $x = 1$. Can you see why that is so?

Solution:

$$\lim_{x\to(-1)^-} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to(-1)^-} \frac{-(6x+3)-3}{x+1} = -6,$$

$$\lim_{x\to(-1)^+} \frac{f(x)-f(-1)}{x-(-1)} = \lim_{x\to(-1)^+} \frac{3x^2-3}{x+1} = \lim_{x\to(-1)^+} 3(x-1) = -6.$$

Hence f is differentiable at $x = -1$ with $f^{'}(-1) = -6$. However, f is not differentiable at $x = 1$ as

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{3x-3}{x-1} = 3, \text{ while}$$

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{3x^2-3}{x-1} = \lim_{x\to 1^-} 3x+3 = 6.$$

Therefore, we have

$$f^{'}(x) = \begin{cases} -6 & \text{if } x \le -1 \\ 6x & \text{if } -1 < x < 1 \\ 3 & \text{if } x > 1 \end{cases}$$

.

(10) Let

$$f(x) = \begin{cases} 4x - 7x^2 & \text{for } x < 0\ , \\ 8x^2 + 4x & \text{for } x \ge 0\ . \end{cases}$$

According to the definition of the derivative, to compute $f'(0)$, we need to compute the left-hand limit
$\lim_{x\to 0^-}$ ____________, which is _____,
and the right-hand limit
$\lim_{x\to 0^+}$ ____________, which is _____.
We conclude that $f'(0)$ is _____

Note: If a limit or derivative is undefined, enter 'undefined' as your answer.

Solution:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^-}\frac{4x-7x^2}{x}=\lim_{x\to 0^-}(4-7x)=4.$$
$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to 0^+}\frac{8x^2+4x}{x}=\lim_{x\to 0^+}(8x+4)=4.$$

Hence $f^{'}(0)=4$.

(11) **Part 1: Limit of a difference quotient**

Suppose $f(x)=\dfrac{3}{x-2}$. Evaluate the limit by using algebra to simplify the difference quotient (in first answer box) and then evaluating the limit (in the second answer box).

$$\lim_{h\to 0}\left(\frac{f(7+h)-f(7)}{h}\right)=\lim_{h\to 0}\left(\rule{3cm}{0.4pt}\right)=\rule{1cm}{0.4pt}.$$

Part 2: Interpreting the limit of a difference quotient

Solution:

$$\lim_{h\to 0}\frac{f(7+h)-f(7)}{h}=\lim_{h\to 0}\frac{\frac{3}{7+h-2}-\frac{3}{5}}{h}=\lim_{x\to 0}\frac{-3}{5(5+h)}=-\frac{3}{25}.$$

The limit of the difference quotient, from Part 1 above, is the value of $f^{'}(7)$, the slope of the tangent line to the graph of $y=f(x)$ at $x=7$ and the instantaneous rate of change of f at $x=7$.

(12) **Part 1: The derivative at a specific point**

Use the definition of the derviative to compute the derivative of $f(x)=\sqrt{x+4}$ at the specific point $x=2$. Evaluate the limit by using algebra to simplify the difference quotient (in first answer box) and then evaluating the limit (in the second answer box).

$$f'(2) = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right) = \lim_{h\to 0}\left(\text{\_\_\_\_\_\_\_\_\_\_}\right) = \_\_\_.$$

Hint: use the *conjugate trick.*

Part 2: The derivative function

Part 3: The tangent line

Solution:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(\sqrt{x+h+4}-\sqrt{x+4})(\sqrt{x+h+4}+\sqrt{x+4})}{h(\sqrt{x+h+4}+\sqrt{x+4})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h+4}+\sqrt{x+4}} = \frac{1}{2\sqrt{x+4}}.$$

Hence $f'(2) = \frac{1}{2\sqrt{6}}$ and $f'(4) = \frac{1}{4\sqrt{2}}$. The tangent line to f at $x = 4$ passes through the point $(4, f(4)) = (4, 2\sqrt{2})$ on the graph of f and the equation for the tangent line to f at $x = 4$ is $y = 2\sqrt{2} + \frac{1}{4\sqrt{2}}(x-4)$.

(13) A function $f(x)$ is said to have a **jump** discontinuity at $x = a$ if:

1. $\lim_{x\to a^-} f(x)$ exists.

2. $\lim_{x\to a^+} f(x)$ exists.

3. The left and right limits are not equal.

Let $f(x) = \begin{cases} 5x-6, & \text{if } x < 9 \\ \frac{2}{x+5}, & \text{if } x \geq 9 \end{cases}$

Show that $f(x)$ has a jump discontinuity at $x = 9$ by calculating the limits from the left and right at $x = 9$.

$\lim_{x\to 9^-} f(x) =$_______

$\lim_{x\to 9^+} f(x) =$_______

Now, for fun, try to graph $f(x)$.

Solution:

We have $\lim_{x\to 9^-} f(x) = 39$ while $\lim_{x\to 9^+} f(x) = \dfrac{1}{7}$.

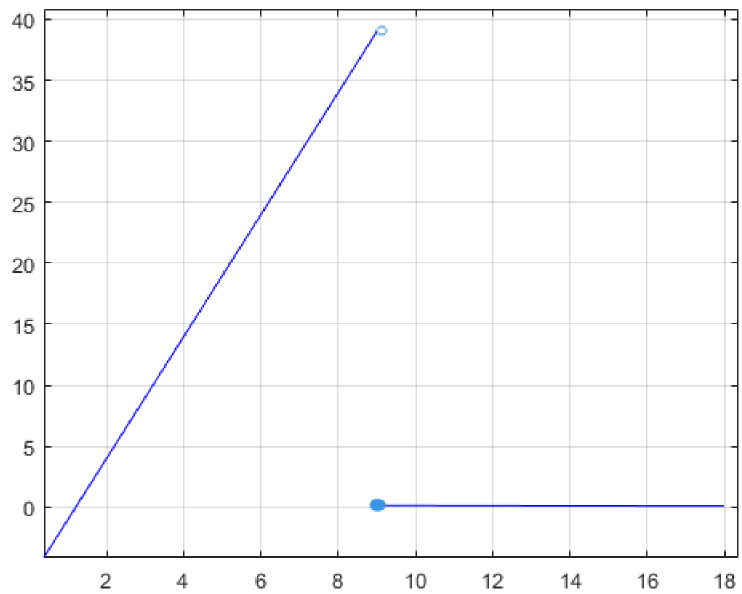
(14) Suppose that

$$f(x+h) - f(x) = 4hx^2 - 3hx + 7h^2x + 8h^2 + 5h^3.$$

Find $f'(x)$.

$f'(x) =$ ___________

Solution



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 4x^2 - 3x.$$

(15) Use the definition of a derivative to find $f'(x)$ and $f'(0)$ where:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

If the derivative does not exist enter DNE.

Solution:

Using the definition of the derivative we find that:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} h^2 \sin\left(\frac{1}{h}\right) \frac{1}{h}$$

$$= \lim_{h \to 0} h \sin(\frac{1}{h})$$

$= 0$, by the Squeeze Theorem.

For $x \neq 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 \sin\frac{1}{x+h} - x^2 \sin\frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(x^2 (\sin \frac{1}{x+h} - \sin \frac{1}{x}) + 2hx \sin \frac{1}{x+h} + h^2 \sin \frac{1}{x+h} \right)$$

The second term will tend to $2x\frac{1}{\sin x}$ and the third term will tend to 0 by the Squeeze Theorem. For the treatment of the first term, recall the facts that

$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2} \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1$$

By using these facts, we have, for $x \neq 0$,

$$\lim_{h\to 0}\frac{x^2}{h}(\sin\frac{1}{x+h} - \sin\frac{1}{x}) = \lim_{h\to 0}\frac{2x^2}{h}\cos\frac{2x+h}{2x(x+h)}\sin\frac{-h}{2x(x+h)}$$

$$= \cos\frac{1}{x}\lim_{h\to 0}\frac{2x^2}{h}\sin\frac{-h}{2x(x+h)}$$

$$= \cos\frac{1}{x}\lim_{h\to 0}\left(\frac{2x^2}{-2x(x+h)}\frac{\sin\frac{-h}{2x(x+h)}}{\frac{-h}{2x(x+h)}}\right)$$

$$= (\cos\frac{1}{x})(-1)(1) = -\cos\frac{1}{x}$$

Therefore we have

$$f^{'}(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \text{ for } x \neq 0.$$

(16) Let

$$f(x) = \begin{cases} -4x^2 + 4x & \text{for } x < 0, \\ 3x^2 - 5 & \text{for } x \geqslant 0. \end{cases}$$

According to the definition of the derivative, to compute $f'(0)$, we need to compute the left-hand limit

$\lim_{x\to 0^-}$ ____________, which is _____,

and the right-hand limit

$\lim_{x\to 0^+}$ ____________, which is _____.

We conclude that $f'(0)$ is _____.

Note: If a limit or derivative does not exist, enter 'DNE' as your answer.

Solution:

$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^-}\frac{-4x^2+4x+5}{x} = \lim_{x\to 0^-}(4+\frac{5}{x})$. Hence this limit does not exist.

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^+}\frac{3x^2-5+5}{x} = \lim_{x\to 0^+}(3x) = 0.$$

Hence f is not differentiable at $x = 0$.

(17) Enter a letter and a number for each formula below so as to define a continuous function. The letter refers to the list of equations and the number is the value of the function f at 1.

___ ___ $\dfrac{|x-1|}{x-1}$ when $x < 1$

___ ___ $\dfrac{1-\cos(x\pi)}{x+1}$ when $x < 1$

___ ___ $\dfrac{\sin(2x-2)}{x-1} + 1$ when $x < 1$

___ ___ $(x-1)\sin\left(\dfrac{1}{x}\right)$ when $x < 1$

A. $-x^2 + 4$ when $x > 1$
B. $\dfrac{\cos(x-1)-1}{x^2}$ when $x > 1$
C. 1 when $x > 1$
D. $\dfrac{|1-x|}{1-x}$ when $x > 1$

Solution:

The answers are D -1; C 1; A 3; B 0. We need to calculate the limits of the above functions at given points and then make pairs.

$$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = -1 = \lim_{x\to 1^+} \frac{|1-x|}{1-x}.$$

$$\lim_{x\to 1^-} \frac{1-\cos(x\pi)}{x+1} = 1.$$

$\displaystyle\lim_{x\to 1^-} \frac{\sin(2x-2)}{x-1} + 1 = 1 + 2\lim_{x\to 1^-} \frac{\sin(2x-2)}{2x-2} = 3 = \lim_{x\to 1^+}(-x^2+4)$. Here we use the fact that $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

$\displaystyle\lim_{x\to 1^-} (x-1)\sin\left(\frac{1}{x}\right) = 0 = \lim_{x\to 1^+} \frac{\cos(x-1)-1}{x^2}$ by the Squeeze Theorem.

(18) Find the value of the constant a that makes the following function continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} \dfrac{6x^3+5x^2+7x+8}{x+1} & \text{if } x < -1 \\ 5x^2 - x + a & \text{if } x \geq -1 \end{cases}$$

$a =$ ____________

Solution:

By simplifying the expression of f, we have $f(x) = 6x^2 - x + 8$ for $x < -1$. Hence we have,

$$\lim_{x\to -1^-} 6x^2 - x + 8 = 15 \text{ and } \lim_{x\to -1^+} 5x^2 - x + a = 6 + a.$$

Hence f is continuous if and only if $15 = 6 + a$, which is $a = 9$.

(19) A function $f(x)$ is said to have a **removable** discontinuity at $x = a$ if:
1. f is either not defined or not continuous at $x = a$.
2. $f(a)$ could either be defined or redefined so that the new function is continuous at $x = a$.

Let $f(x) = \begin{cases} \frac{7}{x} + \frac{-6x+7}{x(x-1)}, & \text{if } x \neq 0, 1 \\ 3, & \text{if } x = 0 \end{cases}$

Show that $f(x)$ has a removable discontinuity at $x = 0$ and determine what value for $f(0)$ would make $f(x)$ continuous at $x = 0$.
Must redefine $f(0) =$______.
Hint: Try combining the fractions and simplifying.
The discontinuity at $x = 1$ is not a removable discontinuity, just in case you were wondering.

Solution:

$$f(x) = \frac{7}{x} + \frac{-6x+7}{x(x-1)} = \frac{1}{x-1} \text{ for } x \neq 0, 1.$$

Hence we must redefine $f(0) = -1$ in order to make f continuous at $x = 0$.

Optional Experimental Problem. Please let us know if you encounter any errors.
This is experimental programming, so it is possible that your answer is correct and well-justified yet a negative feedback message is nonetheless generated.

Please use 'sqrt(x)' to denote $\sqrt{x}$.
Please simplify fractional expressions as much as possible. For example, simplify $\frac{\left(\frac{a}{b}\right)}{c}$ to $\frac{a}{bc}$.

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(20) Let $f(x) = \sqrt{x-2}$, $x > 2$. Use the limit definition of the derivative to find $f'(x)$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h-2}-\sqrt{x-2})(\sqrt{x+h-2}+\sqrt{x-2})}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h-2}+\sqrt{x-2}} = \frac{1}{2\sqrt{x-2}}.$$

(21) Let $f(x) = e^{2x}$. Use the limit definition of the derivative to find $f'(x)$.

Solution:
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{e^{2(x+h)}-e^{2x}}{h} = 2e^{2x} \lim_{h\to 0} \frac{e^{2h}-1}{2h} = 2e^{2x}.$$

Here we use the fact that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.