THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Suggested Solution of Coursework 1

(1) In each part, find a formula for the general term of the sequence, starting with $n = 1$.

(a) $1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$

(b) $1, -\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \ldots$

(c) $\frac{3}{4}, \frac{15}{16}, \frac{63}{64}, \frac{255}{256} \ldots$

(d) $0, \frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \ldots$

Solution:

(a) The general term of the sequence is $a_n = \frac{1}{4^{n-1}}$.

(b) The general term of the sequence is $a_n = \frac{(-1)^{n+1}}{4^{n-1}}$.

(c) The general term of the sequence is $a_n = 1 - \frac{1}{4^n}$.

(d) The general term of the sequence is $a_n = \frac{(n-1)^2}{\sqrt[n]{\pi}}$.

(2) Determine whether the following sequences converge or diverge.

(a) $\{0, 6, 0, 0, 6, 0, 0, 0, 6, \ldots\}$

(b) $a_n = \frac{\sin 2n}{1 + \sqrt{n}}$

(c) $a_n = \frac{n^n}{n!}$

Solution:

(a) The sequence diverges because both 0 and 6 appear indefinitely in the tail of the sequence.

(b) Note that, for all $n \geq 1$, $-1 \leq \sin 2n \leq 1$, and hence

$$-\frac{1}{1+\sqrt{n}} \leq \frac{\sin 2n}{1+\sqrt{n}} \leq \frac{1}{1+\sqrt{n}}.$$

Also, we have $\lim_{n\to\infty} -\frac{1}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$.

By squeeze theorem, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin 2n}{1+\sqrt{n}} = 0$. The sequence converges.

(c) Note that, for all $n \geq 1$,

$$a_n = \frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n} \geq n \cdot 1 \cdots 1 = n.$$

The sequence is unbounded and hence diverges.

(3) Determine whether the sequence $a_n = \frac{1^3}{n^4} + \frac{2^3}{n^4} + \cdots + \frac{n^3}{n^4}$ converges or diverges. If it converges, find the limit.

Solution: Note that, for all $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1^3}{n^4} + \frac{2^3}{n^4} + \cdots + \frac{n^3}{n^4} = \frac{1}{n^4} \sum_{i=1}^{n} i^3 \\ &= \frac{1}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= \frac{1}{4}\left(1 + \frac{1}{n}\right)^2. \end{aligned}$$

Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}(1+0)^2 = \frac{1}{4}$, and the sequence converges.

(4) Determine whether the sequence $a_n = \frac{n^{12} + \sin(13n+8)}{n^{13}+8}$ converges or diverges. If it converges, find the limit.

Solution: Note that, for all $n \geq 1$, $-1 \leq \sin(13n+8) \leq 1$, and hence

$$-\frac{1}{n^{13}+8} \leq \frac{\sin(13n+8)}{n^{13}+8} \leq \frac{1}{n^{13}+8}.$$

Also, we have $\lim_{n\to\infty} -\frac{1}{n^{13}+8} = \lim_{n\to\infty} \frac{1}{n^{13}+8} = 0$.

By squeeze theorem, $\lim_{n\to\infty} \frac{\sin(13n+8)}{n^{13}+8} = 0$. Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{12}}{n^{13}+8} + \lim_{n\to+\infty} \frac{\sin(13n+8)}{n^{13}+8} = \lim_{n\to\infty} \frac{1/n}{1+8/n^{13}} + 0 = 0.$$

(5) Put the following statements in order to justify why $\lim_{n\to\infty} \frac{2n-8+8n^2}{5n^2-5n-8} = \frac{8}{5}$

Solution:

$$\lim_{n\to\infty} \frac{2n-8+8n^2}{5n^2-5n-8} = \frac{8}{5} = \lim_{n\to\infty} \frac{n^2\left(8+\dfrac{2}{n}-\dfrac{8}{n^2}\right)}{n^2\left(5-\dfrac{5}{n}-\dfrac{8}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{8+\dfrac{2}{n}-\dfrac{8}{n^2}}{5-\dfrac{5}{n}-\dfrac{8}{n^2}}$$

$$= \frac{\lim\limits_{n\to\infty}\left(8+\dfrac{2}{n}-\dfrac{8}{n^2}\right)}{\lim\limits_{n\to\infty}\left(5-\dfrac{5}{n}-\dfrac{8}{n^2}\right)}$$

$$= \frac{\lim\limits_{n\to\infty}(8)+2\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right)-8\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}\right)}{\lim\limits_{n\to\infty}(5)-5\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right)-8\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}\right)}$$

$$= \frac{8+2\cdot 0-8\cdot 0}{5-5\cdot 0-8\cdot 0}$$

$$= \frac{8}{5}.$$

(6) Use algebra to simplify the expression before evaluating the limit. In particular, factor the highest power of n from the numerator and denominator, then cancel as many factors of n as possible.

$$\lim_{n\to\infty} \frac{7n}{(6n^3+2)^{1/3}}$$

Solution:

$$\lim_{n\to\infty} \frac{7n}{(6n^3+2)^{1/3}} = \lim_{n\to\infty} \frac{7n}{n(6+2/n^3)^{1/3}} = \lim_{n\to\infty} \frac{7}{(6+2/n^3)^{1/3}} = \frac{7}{6^{1/3}}.$$

(7) Part 1: Evaluating a series

Consider the sequence $\{a_n\} = \left\{\dfrac{2}{n^2+2n}\right\}$.

(a) Find $\lim\limits_{n\to\infty} a_n$ if it exists.

(b) Find $\sum\limits_{n=1}^{\infty} a_n$ if it exists.

Solution:

(a) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2}{n^2+2n} = \lim\limits_{n\to\infty} \dfrac{2/n^2}{1+2/n} = \dfrac{0}{1+0} = 0.$

(b) Note that, for $n \geq 1$,

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Hence, for $N \geq 2$,

$$\begin{aligned}\sum_{n=1}^{N} a_n &= \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+2}\right)\\ &= \left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{N}-\frac{1}{N+2}\right)\\ &= 1+\frac{1}{2}+\left(\frac{1}{3}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{N}-\frac{1}{N}\right)-\frac{1}{N+1}-\frac{1}{N+2}\\ &= \frac{3}{2}-\frac{1}{N+1}-\frac{1}{N+2}.\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty}\sum_{n=1}^{N} a_n = \lim_{N\to\infty}\left(\frac{3}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right)=\frac{3}{2}.$$

Part 2: Evaluating another series

Consider the sequence $\{b_n\} = \left\{\ln\left(\frac{n+1}{n}\right)\right\}$.

(a) Find $\lim_{n\to\infty} b_n$ if it exists.

(b) Find $\sum_{n=1}^{\infty} b_n$ if it exists.

Solution:

(a) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(1+0) = 0.$

(b) Note that, for $n \geq 1$,

$$b_n = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n.$$

Hence, for $N \geq 1$,

$$\begin{aligned}\sum_{n=1}^{N} b_n &= \sum_{n=1}^{N}(\ln(n+1)-\ln n)\\ &= \sum_{n=1}^{N}\ln(n+1) - \sum_{n=1}^{N}\ln n\\ &= \sum_{n=2}^{N+1}\ln n - \sum_{n=1}^{N}\ln n\\ &= \ln(N+1)-\ln 1\\ &= \ln(N+1)\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty}\sum_{n=1}^{N} b_n = \lim_{N\to\infty}\ln(N+1) = +\infty.$$

Part 3: Developing conceptual understanding

Suppose $\{c_n\}$ is a sequence.

Solution:

(a) If $\lim_{n\to\infty} c_n = 0$, then the series $\sum_{n=1}^{\infty} c_n$ **may or may not** converge.
(b) If $\lim_{n\to\infty} c_n \neq 0$, then the series $\sum_{n=1}^{\infty} c_n$ **cannot** converge.
(c) If the series $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n$ **must** be equal to 0.

(8) Consider the sequence

$$a_n = \frac{1}{7n^2 - 896n + 28677}.$$

This sequence is **not** monotone for all n. Determine if the sequence is eventually monotone for $n \geq N$ for some whole number N.

Solution: Note that

$$f(x) = 7x^2 - 896x + 28677 = 7(x-64)^2 + 5 > 0 \quad \text{for any } x \in \mathbb{R}.$$

Hence $a_n = 1/f(n) > 0$ for $n \geq 1$. Moreover, we have

$$\begin{aligned} a_n - a_{n+1} &= \frac{1}{7n^2 - 896n + 28677} - \frac{1}{7(n+1)^2 - 896(n+1) + 28677} \\ &= \frac{14n - 889}{f(n)f(n+1)} \\ &\begin{cases} < 0 & \text{if } 1 \leq n \leq 63 \\ > 0 & \text{if } n \geq 64 \end{cases}. \end{aligned}$$

Therefore the sequence is eventually monotone decreasing, and $N = 64$ is the least value of N such that the sequence is monotone decreasing for $n \geq N$.

(9) Consider the recursively defined sequence:

$$a_1 = 5$$
$$a_{n+1} = \frac{n+1}{n^2} a_n, \quad \text{for } n \geq 1$$

Solution:
(a) Clearly $a_n \geq 0$ for all $n \geq 1$. So the sequence is bounded below by 0.
(b) The sequence is eventually monotone decreasing since $a_2 = 10 > 5 = a_1$ while

$$a_{n+1} = \frac{n+1}{n^2} a_n \leq a_n \quad \text{for } n \geq 2.$$

(c) From above, we see that $a_n \leq a_2 = 10$ for $n \geq 1$. So the sequence is bounded above by 10.
(d) By Monotone Convergence Theorem, $\{a_n\}$ converges to some limit ℓ. Thus

$$\begin{aligned} \ell = \lim_{n\to\infty} a_{n+1} &= \left(\lim_{n\to\infty} \frac{n+1}{n^2}\right)\left(\lim_{n\to\infty} a_n\right) \\ &= \left(\lim_{n\to\infty} (\frac{1}{n} + \frac{1}{n^2})\right) \ell \\ &= 0 \cdot \ell = 0. \end{aligned}$$

Therefore the limit of the sequence $\{a_n\}$ is 0.

(10) Find the following limit.

$$\lim_{n\to\infty} [e^{-5n} \cos(5n)]$$

Solution: Note that, for all $n \geq 1$, $-1 \leq \cos(5n) \leq 1$, and hence

$$-e^{-5n} \leq e^{-5n} \cos(5n) \leq e^{-5n}.$$

Also, we have $\lim_{n\to\infty} -e^{-5n} = \lim_{n\to\infty} e^{-5n} = 0$.
By squeeze theorem, $\lim_{n\to\infty} [e^{-5n} \cos(5n)] = 0$.

(11) Consider the recursively defined sequence:

$$a_1 = \sqrt{7}$$

$$a_{n+1} = \sqrt{7 + a_n}, \quad \text{for } n \geq 1$$

Solution:

(a) The sequence is monotone increasing.
To see this let $Q(n)$ be the statement "$a_{n+1} \geq a_n$".
- When $n = 1$, $a_2 = \sqrt{7 + \sqrt{7}} \geq \sqrt{7} = a_1$. Therefore $Q(1)$ is true.
- Suppose $Q(n)$ is true for some natural number $n \geq 1$, i.e. $a_{n+1} \geq a_n$. Then,

$$a_{n+2} \geq \sqrt{7 + a_{n+1}} \geq \sqrt{7 + a_n} = a_{n+1}.$$

Therefore, $Q(n + 1)$ is true.

By mathematical induction, $a_{n+1} \geq a_n$ for all natural numbers n. Hence $\{a_n\}$ is monotone increasing.

(b) The sequence is bounded below by 0 and bounded above by 4.
To see this, let $P(n)$ be the statement "$0 \leq a_n \leq 4$".
- When $n = 1$, $0 \leq a_1 = \sqrt{7} \leq 4$. Therefore $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n \geq 1$, i.e. $0 \leq a_n \leq 4$. Then,

$$0 \leq a_{n+1} = \sqrt{7 + a_n} \leq \sqrt{7 + 4} = \sqrt{11} \leq 4.$$

Therefore, $P(n + 1)$ is true.

By mathematical induction, $0 \leq a_n \leq 4$ for all natural numbers n. Hence $\{a_n\}$ is bounded.

(c) By Monotone Convergence Theorem, $\{a_n\}$ is convergent. Let $\lim_{n\to\infty} a_n = A$. Since $a_{n+1}^2 = 7 + a_n$, we have

$$\lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (7 + a_n)$$
$$A^2 = 7 + A$$
$$A^2 - A - 7 = 0.$$

So $A = \dfrac{1 + \sqrt{29}}{2}$ or $A = \dfrac{1 - \sqrt{29}}{2}$, where the later is rejected since $a_n \geq 0$.
Therefore, $\lim_{n\to\infty} a_n = \dfrac{1 + \sqrt{29}}{2}$.

(12) Consider the recursively defined sequence:

$$a_1 = 1, \quad a_2 = 1$$
$$a_{n+2} = \frac{a_{n+1} + a_n}{2}, \qquad \text{for } n \geq 1$$

Find the limit of the sequence if it exists.

Solution:

From the definition of the sequence,

$$a_3 = \frac{a_2 + a_1}{2} = \frac{1+1}{2} = 1,$$
$$a_4 = \frac{a_3 + a_2}{2} = \frac{1+1}{2} = 1,$$

and so on, we thus have

$$a_{n+2} = \frac{a_{n+1} + a_n}{2} = \frac{1+1}{2} = 1, \qquad \text{for } n \geq 1.$$

Hence the sequence is just a constant sequence of 1's, and clearly $\lim_{n\to\infty} a_n = 1$.

(13) Consider the sequence

$$a_n = \frac{n\cos(n\pi)}{2n-1}.$$

Write the first five terms of a_n, and find $\lim_{n\to\infty} a_n$.

Solution: The first five terms are

$$a_1 = -1,\ a_2 = \frac{2}{3},\ a_3 = -\frac{3}{5},\ a_4 = \frac{4}{7},\ a_5 = -\frac{5}{9}.$$

Note that

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2n\cos(2n\pi)}{4n-1} = \lim_{n\to\infty} \frac{1}{2 - 1/2n} = \frac{1}{2},$$

while

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{(2n+1)\cos((2n+1)\pi)}{4n+1} = \lim_{n\to\infty} -\frac{1+1/2n}{2+1/2n} = -\frac{1}{2}.$$

Since $\lim_{n\to\infty} a_{2n} \neq \lim_{n\to\infty} a_{2n+1}$, $\lim_{n\to\infty} a_n$ does not exist.

(14) The sequence $\{a_n\}$ is defined by $a_1 = 2$, and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

for $n \geq 1$. Assuming that $\{a_n\}$ converges, find its limit.

Solution: Let $a = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we have

$$a = \frac{1}{2}\left(a + \frac{2}{a}\right)$$
$$2a^2 = a^2 + 2$$
$$a^2 = 2.$$

So $a = \sqrt{2}$ or $a = -\sqrt{2}$, where the later is rejected since $a_n \geq 0$. Therefore, $\lim_{n\to\infty} a_n = a = \sqrt{2}$.

(15) Determine whether the sequence is divergent or convergent. If it is convergent, evaluate its limit.

$$\lim_{n\to\infty} \frac{18 + 25\arctan(n!)}{19^n}$$

Solution: Note that, for all $n \geq 1$,

$$-\frac{\pi}{2} \leq \arctan(n!) \leq \frac{\pi}{2},$$

and hence

$$\frac{18 - \frac{25\pi}{2}}{19^n} \leq \frac{18 + 25\arctan(n!)}{19^n} \leq \frac{18 + \frac{25\pi}{2}}{19^n}$$

Also, we have $\lim\limits_{n\to\infty} \frac{18 - \frac{25\pi}{2}}{19^n} = \lim\limits_{n\to\infty} \frac{18 + \frac{25\pi}{2}}{19^n} = 0$.

By squeeze theorem, $\lim\limits_{n\to\infty} \frac{18 + 25\arctan(n!)}{19^n} = 0$.

(16) Determine whether the sequence is divergent or convergent. If it is convergent, evaluate its limit.

$$\lim_{n\to\infty} (-1)^n \sin(4/n)$$

Solution: Note that, for $n \geq 1$,

$$-|\sin(4/n)| \leq (-1)^n \sin(4/n) \leq |\sin(4/n)|.$$

Moreover, $\lim\limits_{n\to\infty} |\sin(4/n)| = |\sin(0)| = 0$, and similarly $\lim\limits_{n\to\infty} -|\sin(4/n)| = 0$. Therefore $\lim\limits_{n\to\infty} (-1)^n \sin(4/n) = 0$.

(17) Consider the sequence

$$a_n = \frac{(3n-1)!}{(3n+1)!}.$$

Describe the behaviour of the sequence.

Solution: Note that $a_n = \frac{1}{3n(3n+1)}$. Thus, for $n \geq 1$,

$$a_n = \frac{1}{3n(3n+1)} \geq \frac{1}{3(n+1)(3(n+1)+1)} = a_{n+1}$$

Hence $\{a_n\}$ is monotone decreasing for all n.

Moreover

$$0 \leq a_n = \frac{1}{3n(3n+1)} \leq 1, \qquad \text{for all } n \geq 1.$$

So $\{a_n\}$ is both bounded above and bounded below.

By Monotone Convergence Theorem, $\{a_n\}$ converges. Indeed,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n(3n+1)} = 0.$$