THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Fall 2020)
Suggested Solution of Coursework 10

1. Evaluate the integral

$$\int_0^{\pi/3} \frac{8\sin(x) + 8\sin(x)\tan^2(x)}{\sec^2(x)} dx$$

Solution:

$$\int_0^{\pi/3} \frac{8\sin(x) + 8\sin(x)\tan^2(x)}{\sec^2(x)} dx$$
$$= \int_0^{\pi/3} \frac{8\sin(x)\sec^2(x)}{\sec^2(x)} dx$$
$$= \int_0^{\pi/3} 8\sin(x) dx$$
$$= -8\cos(x)\Big|_0^{\pi/3}$$
$$= 4$$

2. Write the integral as a sum of integrals without absolute values and evaluate:

$$\int_{\pi/4}^{\pi} |\cos(x)| dx$$

Solution:

$$\int_{\pi/4}^{\pi} |\cos(x)| dx$$
$$= \int_{\pi/4}^{\pi/2} \cos(x) dx - \int_{\pi/2}^{\pi} \cos(x) dx$$
$$= \sin(x)\Big|_{\pi/4}^{\pi/2} - \sin(x)\Big|_{\pi/2}^{\pi}$$
$$= \left(1 - \frac{\sqrt{2}}{2}\right) - (0 - 1)$$
$$= 2 - \frac{\sqrt{2}}{2}$$

3. Evaluate the integral

$$\int_0^4 |\sqrt{x+2} - x| dx$$

Solution:

$$\int_0^4 |\sqrt{x+2} - x| dx$$

$$= \int_0^2 (\sqrt{x+2} - x)dx - \int_2^4 (\sqrt{x+2} - x)dx$$
$$= \left(\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2\right)_0^2 - \left(\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2\right)_2^4$$
$$= \left[\left(\frac{16}{3} - 2\right) - \frac{4}{3}\sqrt{2}\right] - \left[\left(4\sqrt{6} - 8\right) - \left(\frac{16}{3} - 2\right)\right]$$
$$= \frac{44}{3} - 4\sqrt{6} - \frac{4}{3}\sqrt{2}$$

4. Evaluate the definite integral

$$\int_{-4}^{5} (x - 4|x|)dx$$

Solution:

$$\int_{-4}^{5} (x - 4|x|)dx$$
$$= \int_{-4}^{0} 5xdx - \int_0^5 3xdx$$
$$= \frac{5}{2}x^2\Big|_{-4}^{0} - \frac{3}{2}x^2\Big|_0^5$$
$$= -40 - \frac{75}{2}$$
$$= -\frac{155}{2}$$

5. Find the area between the curves:

$y = x^3 - 14x^2 + 40x$

and $y = -x^3 + 14x^2 - 40x$

Solution:

Solving for $y = x^3 - 14x^2 + 40x = 0$, $x = 0, 4, 10$.

So the area

$$= 2\int_0^4 (x^3 - 14x^2 + 40)dx - 2\int_4^{10} (x^3 - 14x^2 + 40)dx$$
$$= 2\left(\frac{1}{4}x^4 - \frac{14}{3}x^3 + 40x\right)_0^4 - 2\left(\frac{1}{4}x^4 - \frac{14}{3}x^3 + 40x\right)_4^{10}$$
$$= \frac{2024}{3}$$

6. Consider the area between the graphs $x + 3y = 19$ and $x + 9 = y^2$. This area can be computed in two different ways using integrals.

First of all it can be computed as a sum of two integrals

$$\int_a^b f(x)dx + \int_b^c g(x)dx$$

where $a =$ _, $b =$ _, $c =$ _ and

$f(x) =$ _

$g(x) =$ _

Alternatively this area can be computed as a single integral

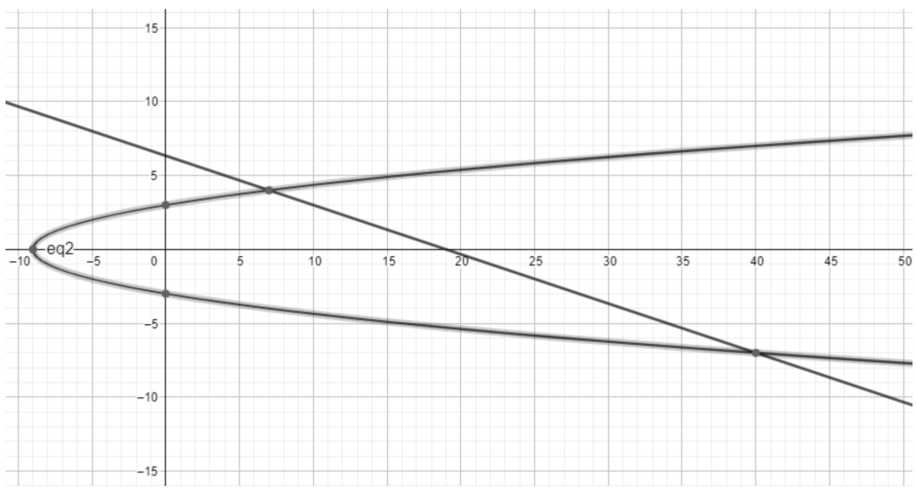
$$\int_{\alpha}^{\beta} h(y)dy$$

where $\alpha =$ _, $\beta =$ _ and

$h(x) =$ _

Either way we find that the area is _.

Solution:



Setting up simultaneous equation, we find that the two graphs meet at $(7, 4)$ and $(40, -7)$.

Also, the vertex of the parabola is at $(-9, 0)$

Hence, the area

$= 2\int_{-9}^{7} \sqrt{x+9}dx + \int_{7}^{40} \left(\sqrt{x+9} + \frac{1}{3}(19-x)\right) dx$

$= \frac{1331}{6}$

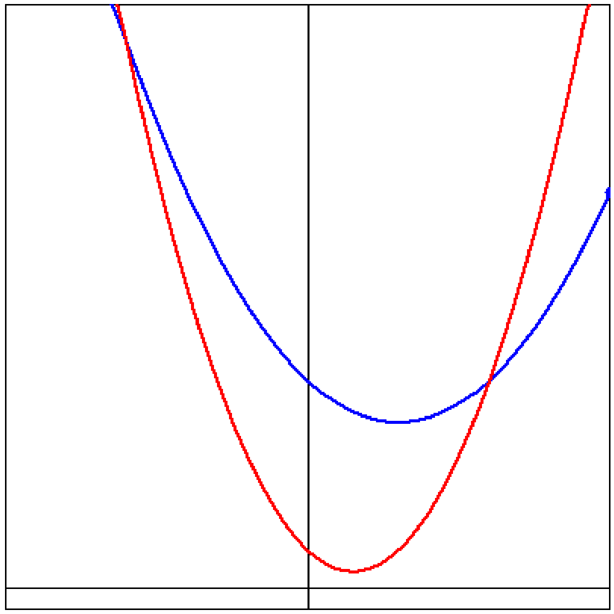
Alternatively, the area

$= \int_{-7}^{4} [(19-3y) - (y^2-9)]dy$

$= -\frac{1}{3}y^3 - \frac{3}{2}y^2 + 28y\Big|_{-7}^{4}$

$= \frac{1331}{6}$

7. Find the area of the region enclosed between $f(x) = x^2 - 3x + 11$ and $g(x) = 2x^2 - 3x + 2$.



Solution:

The points of intersection are $(-3, 29)$ and $(3, 11)$.

Area

$= \int_{-3}^{3} [(x^2 - 3x + 11) - (2x^2 - 3x + 2)]dx$

$= \int_{-3}^{3} (-x^2 + 9)dx$

$= -\frac{1}{3}x^3 + 9x\Big|_{-3}^{3}$

$= 36$

8. Find the area of the region between the curves $y = |x|$ and $y = x^2 - 2$.

Solution:

The points of intersection are $(\pm 2, 2)$.

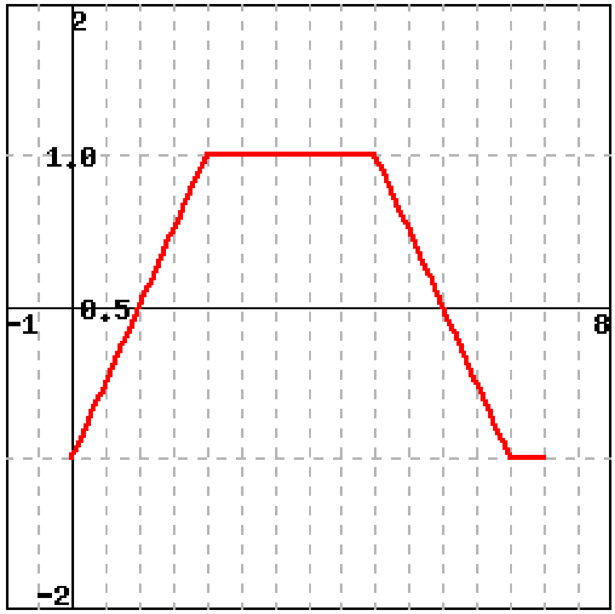
By symmetry, area

$= 2\int_{0}^{2} [x - (x^2 - 2)]dx$

$= 2\left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right)\Big|_{0}^{2}$

$= \frac{20}{3}$

9. Let $A(x) = \int_0^x f(t)dt$, with $f(x)$ as in figure.



$A(x)$ has a local minimum at $x =$ _

$A(x)$ has a local maximum at $x =$ _

Solution:

Local extrema occur when the first derivative is zero.

By Fundamental Theorem of Calculus, the first derivative is exactly $f(x)$.

With second derivative test, we know that:

$A(x)$ has a local minimum at $x = 1$.

$A(x)$ has a local maximum at $x = \frac{11}{2}$

10. $G(x) = \int_1^x \tan t dt$

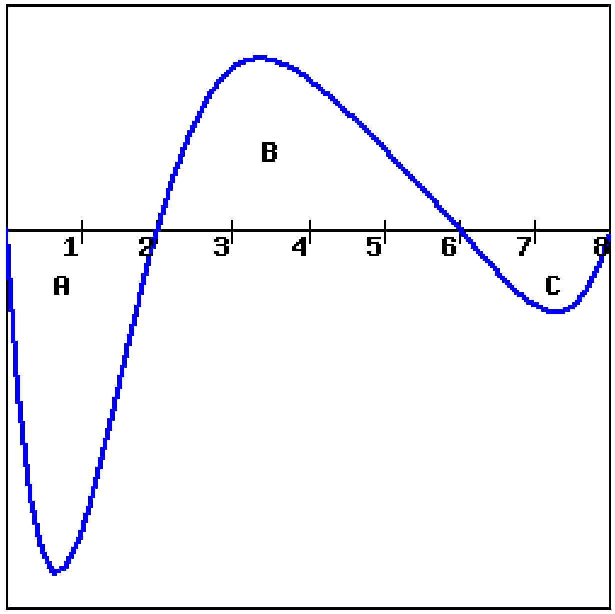
    Find $G(1)$, $G'(\frac{\pi}{5})$.

    Solution:

    $G(1) = 0$

    $G'\left(\frac{\pi}{5}\right) = \tan\left(\frac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}}$

11. The figure below gives $F'(x)$ for some function F.



Use this graph and the facts that the area labeled A is 15, that labeled B is 16, that labeled C is 4, and that $F(2) = -4$ to sketch the graph of $F(x)$. Label the values of at least four points.

Then, using your graph, give four (x, y) points on the curve.

Solution:

By fundamental theorem of calculus, four points are:

$(2, -4)$

$(0, -4 + 15) = (0, 11)$

$(6, -4 + 16) = (6, 12)$

$(8, 12 - 4) = (8, 8)$

12. Given

$$f(x) = \int_0^x \frac{t^2 - 25}{1 + \cos^2(t)} dt$$

At what value of x does the local max of $f(x)$ occur?

Solution:

$f'(x) = \dfrac{x^2 - 25}{1 + \cos^2(x)} = 0$ when $x = \pm 5$.

We also see that:

$f'(x) < 0$ when $-5 < x < 5$

$f'(x) > 0$ when $x < -5$ or $x > 5$

So, $f(x)$ has a local max when $x = -5$.

13. Let

$$f(x) = \begin{cases} 0 & \text{if } x < -3 \\ 2 & \text{if } -3 \leq x < -1 \\ -3 & \text{if } -1 \leq x < 4 \\ 0 & \text{if } x \geq 4 \end{cases}$$

and

$$g(x) = \int_{-3}^{x} f(t)dt$$

Determine the value of each of the following:

(a) $g(-7) =$ _

(b) $g(-2) =$ _

(c) $g(0) =$ _

(d) $g(5) =$ _

(e) The absolute maximum of $g(x)$ occurs when $x =$ _ and is the value _.

Solution:

(a) $g(-7) = 0$

(b) $g(-2) = 2 \times [(-2) - (-3)] = 2$

(c) $g(0) = 2 \times 2 + (-3) \times 1 = 1$

(d) $g(5) = 2 \times 2 + (-3) \times 5 = -11$

(e) The absolute maximum of $g(x)$ occurs when $x = -1$ and is the value 4.

14. (a) Suppose $F(x)$ is any function that is differentiable for all real numbers x. Evaluate the following derivative.

$$\frac{d}{dx}(F(x^4))$$

(b) Suppose $F(x) = \int_{12}^{x} e^{-t^2} dt$. Use the Fundamental Theorem of Calculus to evaluate the derivative.

$$F'(x) = \frac{d}{dx} \int_{12}^{x} e^{-t^2} dt = \_$$

(c) Suppose $F(x) = \int_{12}^{x} e^{-t^2} dt$. Find a formula for the function $F(x^4)$ expressed using an integral.

$$F(x^4) = \_$$

(d)

$$\frac{d}{dx}(F(x^4)) = \frac{d}{dx}\left(\int_{12}^{x^4} e^{-t^2} dt\right) = \_$$

Solution:

(a) By chain rule, $\frac{d}{dx}(F(x^4)) = 4x^3 F'(x^4)$

(b) By fundamental theorem of calculus,
$F'(x) = e^{-x^2}$

(c) $F(x^4) = \int_{12}^{x^4} e^{-t^2} dx$

(d) Combining the above,
$\frac{d}{dx}(F(x^4)) = 4x^3 e^{-x^8}$

15. Use the Fundamental Theorem of Calculus to find the derivative of

$$y = \int_{-9}^{\sqrt{x}} \frac{\cos t}{t^4} dt$$

.

Solution:

By fundamental theorem of calculus and chain rule, we have

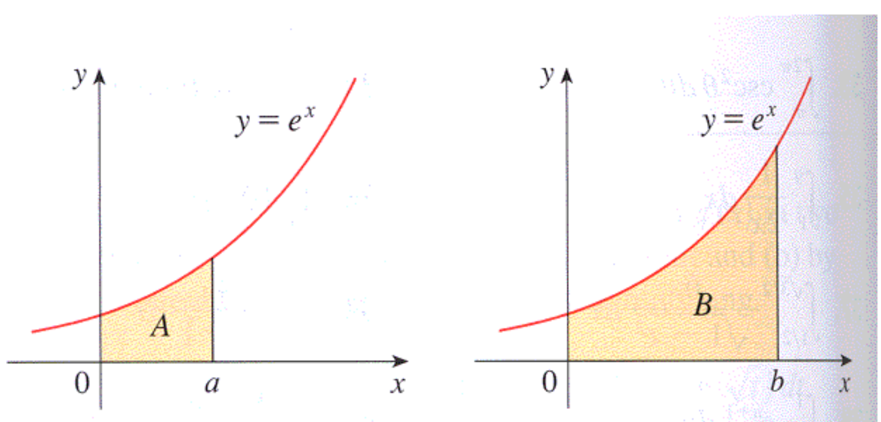
$\frac{dy}{dx}$

$= \frac{d}{dx}\left(\int_{-9}^{\sqrt{x}} \frac{\cos t}{t^4} dt\right)$

$= \frac{\cos\sqrt{x}}{x^2}\frac{d}{dx}(\sqrt{x})$

$= \frac{\cos\sqrt{x}}{2x^2\sqrt{x}}$

16. The area labeled B is 8 times the area labeled A. Express b in terms of a.



Solution:

$$\int_0^b e^x dx = 8\int_0^a e^x dx$$

$$e^b - 1 = 8(e^a - 1)$$

$$b = \ln(8e^a - 7)$$