THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2020-2021 Term 1
Homework Assignment 4
Due Date: 7 December 2020 (Monday)

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Signature Date

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- Late assignments will receive a grade of 0.
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- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Evaluate the following integrals:

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$

(b) $\displaystyle\int \frac{x^5}{(1+x^3)^3}dx$

(c) $\displaystyle\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$

(d) $\displaystyle\int \frac{x-2}{\sqrt{x^2-4x+3}}\,dx$

(e) $\displaystyle\int x^2\sqrt{x^3+2}\,dx$

(f) $\displaystyle\int \sin^5 x\cos x dx$

(g) $\displaystyle\int \sin 3x\sin 5x dx$

(h) $\displaystyle\int \cos x\cos 7x\,dx$

(i) $\displaystyle\int \sin^2 2x\sin 5x\,dx$

(j) $\displaystyle\int \cos^2 2x\sin^3 2x\,dx$

Solution:

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}}dx^2 = \frac{1}{2}\int \frac{1}{\sqrt{1+t}}dt = \sqrt{1+x^2}+C$

(b) $\displaystyle\int \frac{x^5}{(1+x^3)^3}dx = \frac{1}{3}\int \frac{x^3}{(1+x^3)^3}dx^3 = -\frac{1}{6}\int td\frac{1}{(1+t)^2} = -\frac{1}{6}(t\frac{1}{(1+t)^2} - \int \frac{1}{(1+t)^2}dt) = -\frac{1}{6}\frac{2x^3+1}{(1+x^3)^2}+C$

(c) $\displaystyle\int \frac{1}{x^2}\sin\frac{1}{x}dx = -\int \sin\frac{1}{x}d\frac{1}{x} = \cos\frac{1}{x}+C$

(d) $\int \frac{x-2}{\sqrt{x^2-4x+3}}dx = \int \frac{x-2}{\sqrt{(x-1)(x-3)}}dx = \int \frac{t}{\sqrt{t^2-1}}dt = \frac{1}{2}\int \frac{1}{\sqrt{t^2-1}}dt^2 = \sqrt{t^2-1}+C = \sqrt{(x-2)^2-1}+C$

(e) $\displaystyle\int x^2\sqrt{x^3+2}dx = \frac{1}{3}\int \sqrt{x^3+2}dx^3 = \frac{2}{9}(x^3+2)^{\frac{3}{2}}+C$

(f) $\displaystyle\int \sin^5 x\cos x dx = \int \sin^5 x d\sin x = \frac{1}{6}\sin^6 x + C$

(g) $\displaystyle\int \sin 3x\sin 5x\ dx = -\frac{1}{2}\int \cos 8x - \cos 2x\ dx = -\frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$

(h) $\displaystyle\int \cos x\cos 7x\ dx = \frac{1}{2}\int \cos 8x + \cos 6x\ dx = \frac{1}{16}\sin 8x + \frac{1}{12}\sin 6x + C$

(i) $\displaystyle\int \sin^2 2x\sin 5x\ dx = \int \frac{1-\cos 4x}{2}\sin 5x\ dx = \frac{1}{2}\Big(\int \sin(5x)dx - \int \cos(4x)\sin(5x)dx\Big) = -\frac{\cos(5x)}{10} - \frac{1}{4}\int (\sin(5x+4x)+\sin(5x-4x))dx = -\frac{\cos(5x)}{10} - \frac{1}{4}\int (\sin(9x)+\sin(x))dx = -\frac{1}{10}\cos 5x + \frac{1}{36}\cos 9x + \frac{1}{4}\cos x + C$

(j) $\displaystyle\int \cos^2 2x\sin^3 2x\ \ dx = -\frac{1}{2}\int \cos^2 2x(1-\cos^2 2x)\ \ d\cos 2x = -\frac{1}{6}\cos^3 2x + \frac{1}{10}\cos^5 2x + C$

where C is an arbitrary constant.

2. Evaluate the following indefinite integrals.

(a) $\int x^2 \ln x dx$

(b) $\int x \sec^2 x dx$

(c) $\int e^{-x} \sin 3x \, dx$

(d) $\int \sin(\ln x) dx$

Solution:

(a) $\int x^2 \ln x \; dx = \int x^2 \; d(x\ln x - x) = x^2(x\ln x - x) - \int 2(x\ln x - x)x \; dx = x^3 \ln x - x^3 - 2\int x^2 \ln x \; dx + \frac{2}{3}x^3 + C \Rightarrow \int x^2 \ln x \; dx = \frac{1}{3}(x^3 \ln x - \frac{1}{3}x^3) + C$

(b) $\int x \frac{1}{\cos^2 x} = \int x d \tan x = x \tan x - \int \tan x dx = x \tan x + \ln|\cos x| + C$

(c) $\int e^{-x} \sin 3x dx = -\int \sin 3x de^{-x} = -e^{-x} \sin 3x + 3\int e^{-x} \cos 3x dx$
on the other hand,
$\int e^{-x} \cos 3x dx = -\int \cos 3x de^{-x} = -e^{-x} \cos 3x - 3\int e^{-x} \sin 3x dx$
$\Rightarrow \int e^{-x} \sin 3x dx = -\frac{1}{10}(e^{-x} \sin 3x + 3e^{-x} \cos 3x) + C$

(d) Let $t = \ln x$. Then $\int \sin(\ln x) dx = \int \sin t de^t = e^t \sin t - \int e^t \cos t dt$
on the other hand,
$\int e^t \cos t dt = e^t \cos t + \int e^t \sin t dt$
$\Rightarrow$
$\int e^t \sin t = \frac{1}{2}(e^t \sin t - e^t \cos t) = \frac{1}{2}(x \sin(\ln x) - x \cos(\ln x)) + C$

3. Find $F'(x)$ for the following functions.

(a) $F(x) = \int_\pi^x \frac{\cos y}{y} dy$

(b) $F(x) = \int_0^{x^3} e^{u^2} \, du$

(c) $F(x) = \int_x^{2x} (\ln t)^2 \, dt$

(d) $F(x) = \int_{x^2}^{x^3} e^{\cos u} \, du$

(e) $F(x) = \int_1^x \frac{e^x + e^t}{t} dt$

(f) $F(x) = \int_1^x \frac{e^{xt}}{t} dt$

Solution:

(a) $F'(x) = \frac{\cos x}{x}$

(b) $F'(x) = 3x^2 e^{x^6}$

(c) $F(x) = \int_1^{2x} (\ln t)^2 dt + \int_x^1 (\ln t)^2 dt \Rightarrow F'(x) = 2(\ln(2x))^2 - (\ln x)^2$

(d) $F'(x) = 3x^2 e^{\cos x^3} - 2xe^{\cos x^2}$

(e) $F(x) = e^x \int_1^x \frac{1}{t}dt + \int_1^x \frac{e^t}{t}dt \Rightarrow F'(x) = e^x(\ln x + \frac{2}{x})$

(f) Put $u = xt$, then

$$F(x) = \int_1^x \frac{e^{xt}}{t}dt = \int_x^{x^2} \frac{e^u}{u/x} \cdot \frac{1}{x}du = \int_x^{x^2} \frac{e^u}{u}du.$$

Hence

$$F^{'}(x) = 2x \cdot \frac{e^{x^2}}{x^2} - \frac{e^x}{x} = \frac{2e^{x^2}}{x} - \frac{e^x}{x}.$$

4. Evaluate the following definite integrals.

(a) $\int_0^1 x^3\sqrt{1+3x^2}dx$

(b) $\int_0^\pi x \sin 2x \, dx$

(c) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx$

(d) $\int_0^5 |x^2 - 4x + 3| \, dx$

Solution:

(a) $\int_0^1 x^3\sqrt{1+3x^2}dx = \frac{1}{2}\int_0^1 x^2\sqrt{1+3x^2}dx^2 = \frac{1}{2}\int_0^1 td\frac{2}{9}(1+3t)^{\frac{3}{2}} = \frac{1}{9}t(1+3t)^{\frac{3}{2}}|_0^1 - \frac{1}{9}\int_0^1 (1+3t)^{\frac{5}{2}}dx = \frac{58}{135}$

(b) $\int_0^\pi x\sin 2xdx = \int_0^\pi -\frac{1}{2}xd\cos 2x = -\frac{1}{2}(x\cos 2x|_0^\pi - \int_0^\pi \cos 2xdx) = -\frac{\pi}{2}$

(c) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}dx = -\int_0^1 xd\sqrt{4-x^2} = -(x\sqrt{4-x^2}|_0^1 - \int_0^1 \sqrt{4-x^2}dx) = -x\sqrt{4-x^2}|_0^1 + \int_0^1 \frac{4}{\sqrt{4-x^2}}dx - \int_0^1 \frac{x^2}{\sqrt{4-x^2}}dx$

So $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}dx = -\frac{1}{2}x\sqrt{4-x^2}|_0^1 + \frac{4}{2}\int_0^1 \frac{1}{\sqrt{1-(x/2)^2}}d(x/2) = -\frac{\sqrt{3}}{2} + 2\arcsin(\frac{x}{2})\Big|_0^1 = -\frac{\sqrt{3}}{2} + \frac{\pi}{3}$

(d) $\int_0^5 |x^2-4x+3|dx = \int_0^1 (x^2-4x+3)dx - \int_1^3 (x^2-4x+3)dx + \int_3^5 (x^2-4x+3)dx = \frac{28}{3}$

5. Prove the following reduction formulas.

(a) $I_n = \int \sin^n xdx$; $I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}I_{n-2}$, $n \geq 2$

(b) $I_n = \int (\ln x)^n dx$; $I_n = x(\ln x)^n - nI_{n-1}$, $n \geq 1$.

(c) $I_n = \int x^n \cos x dx$; $I_n = x^n \sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}$, $n \geq 2$

(d) $I_n = \int \frac{dx}{(x^2-a^2)^n}$; $I_n = -\frac{x}{2a^2(n-1)(x^2-a^2)^{n-1}} - \frac{2n-3}{2a^2(n-1)}I_{n-1}$, $n \geq 1$

Solution:

(a) $I_n = \int \sin^n x dx = -\int \sin^{n-1} x d\cos x = -(\cos x \sin^{n-1} x - (n-1)\int \cos^2 x \sin^{n-2} dx) = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \Rightarrow I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}I_{n-2}$

(b) $I_n = x(\ln x)^n - n\int x\frac{1}{x}(\ln x)^{n-1}dx = x(\ln x)^n - nI_{n-1}$

(c) $I_n = \int x^n d\sin x = x^n \sin x - n\int x^{n-1}\sin x dx = x^n \sin x + n(x^{n-1}\cos x - (n-1)\int x^{n-2}\cos x dx) = x^n \sin x + nx^{n-1}\cos x - n(n-1)I_{n-2}$

(d) $I_n = \int \frac{dx}{(x^2-a^2)^n} = x\frac{1}{(x^2-a^2)^n} + n\int x\frac{2x}{(x^2-a^2)^{n+1}}dx = x\frac{1}{(x^2-a^2)^n} + 2n\int \frac{dx}{(x^2-a^2)^n} + 2na^2\int \frac{dx}{(x^2-a^2)^{n+1}} = \frac{x}{(x^2-a^2)^n} + 2nI_n + 2na^2 I_{n+1} \Rightarrow$
$I_{n+1} = -\frac{x}{2a^2n(x^2-a^2)^n} - \frac{2n-1}{2a^2n}I_n.$

Hence $I_n = -\frac{x}{2a^2(n-1)(x^2-a^2)^{n-1}} - \frac{2n-3}{2a^2(n-1)}I_{n-1}$

6. Evaluate the following integrals of rational functions.

(a) $\int \frac{x^2 dx}{1-x^2}$

(b) $\int \frac{x^3}{x+3}dx$

(c) $\int \frac{4x+1}{x^2-6x+13}dx$

(d) $\int \frac{(1+x)^2}{1+x^2}dx$

(e) $\int \frac{2x^3-x^2+3}{x^2-2x-3}dx$

(f) $\int \frac{6x+11}{(x+1)^2}dx$

(g) $\int \frac{x^2+1}{(x+1)^2(x-1)}, dx$

(h) $\int \frac{x^4}{x^2-4}dx$

(i) $\int \frac{dx}{(x+1)(x^2+1)}$

(j) $\int \frac{dx}{x(x^2+1)^2}$

Solution:

(a) $\int \frac{x^2 dx}{1-x^2} = \int \left(-1 + \frac{1}{1-x^2}\right) = -x + \frac{1}{2}\int \left(\frac{1}{1-x} + \frac{1}{1+x}\right)$
$= -x + \frac{-\ln|1-x| + \ln|1+x|}{2} + C$

(b) $\int \frac{x^3}{x+3}\,dx = \int \frac{x^3+27-27}{x+3}dx = \int \frac{x^3+27}{x+3}dx - 27\int \frac{1}{x+3}dx$
$= \int (x^2-3x+9)dx - 27\int \frac{1}{x+3}dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln|x+3| + C$

(c) $\int \frac{4x+1}{x^2-6x+13}dx = \int \frac{4x-12+13}{x^2-6x+13}dx = 2\int \frac{2x-6}{x^2-6x+13}dx + 13\int \frac{1}{x^2-6x+13}dx$
$= 2\int \frac{d(x^2-6x+13)}{x^2-6x+13}dx + 13\int \frac{1}{(x-3)^2+4}dx$
$= 2\ln(x^2-6x+13) + 13\cdot\frac{2}{4}\int \frac{1}{(\frac{x-3}{2})^2+1}d(\frac{x-3}{2})$
$= 2\ln(x^2-6x+13) + \frac{13}{2}\arctan(\frac{x-3}{2}) + C$

(d) $\int \frac{(1+x)^2}{1+x^2}\,dx = \int \frac{x^2+1+2x}{1+x^2}dx = \int (1+\frac{2x}{1+x^2})dx = x + \int \frac{d(x^2+1)}{1+x^2}$
$= x + \ln(1+x^2) + C$

(e) $\int \frac{2x^3-x^2+3}{x^2-2x-3}\,dx = \int (2x+3+12\frac{x+1}{(x-3)(x+1)})dx = x^2+3x+12\int \frac{1}{x-3}dx$
$= x^2+3x+12\ln|x-3| + C$

(f) $\int \frac{6x+11}{(x+1)^2}\,dx = \int \frac{6(x+1)+5}{(x+1)^2}\,dx = 6\int \frac{1}{(x+1)}dx + 5\int \frac{1}{(x+1)^2}dx$
$= 6\ln|x+1| - \frac{5}{x+1} + C$

(g) $\int \frac{x^2+1}{(x+1)^2(x-1)},dx = \int \left[\frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{1}{(x+1)^2}\right]dx$
$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + \frac{1}{x+1} + C$

(h) $\int \frac{x^4}{x^2-4}\,dx = \int \frac{x^4-16+16}{x^2-4}\,dx = \int x^2+4+\frac{16}{x^2-4}\,dx$
$= \frac{1}{3}x^3 + 4x + 4\int (\frac{1}{x-2} - \frac{1}{x+2})dx = \frac{1}{3}x^3 + 4x + 4\ln\frac{|x-2|}{|x+2|} + C$

(i) $\int \frac{dx}{(x+1)(x^2+1)} = \int \frac{1}{2}(\frac{1}{x+1} + \frac{-x+1}{x^2+1})dx$
$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x^2+1}dx$
$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\int \frac{d(x^2+1)}{x^2+1} + \frac{1}{2}\int \frac{1}{x^2+1}dx$
$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\arctan x + C$

(j) $\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} - \frac{x^3+2x}{(x^2+1)^2}\right]dx = \ln|x| - \int \frac{x^3+x}{(x^2+1)^2}dx - \int \frac{x}{(x^2+1)^2}dx$
$= \ln|x| - \frac{1}{4}\int \frac{d(x^2+1)^2}{(x^2+1)^2} - \frac{1}{2}\int \frac{d(x^2+1)}{(x^2+1)^2} = \ln|x| - \frac{1}{4}\ln|(x^2+1)^2| + \frac{1}{2}\frac{1}{1+x^2} + C$

7. Evaluate the following integrals by trigonometric substitution.

(a) $\displaystyle\int \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

(c) $\displaystyle\int \frac{2}{x^3\sqrt{x^2-1}}dx$

(d) $\displaystyle\int x^2\sqrt{16-x^2}dx$

Solution:

(a) Put $x = \sin t$, we have

$$\int \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}} = \int \frac{\sin^2 t}{\cos^3 t}\cos tdt = \int \tan^2 tdt = \int (\sec^2 t - 1)dt$$
$$= \tan t - t + C = \frac{x}{\sqrt{1-x^2}} - \arcsin x + C$$

(b) Put $x = 2\tan t, t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then $\sqrt{4+x^2} = 2\sec t, dx = 2\sec^2 tdt$,

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{1}{2\sec t}2\sec^2 tdt = \int \sec tdt = \int \frac{1}{\cos t}dt = \int \frac{\cos t}{\cos^2 t}dt$$
$$= \int \frac{1}{1-\sin^2 t}d(\sin t) = \frac{1}{2}\int \left[\frac{1}{1-\sin t} + \frac{1}{1+\sin t}\right] d\sin t$$
$$= \frac{1}{2}\ln\frac{|1+\sin t|}{|1-\sin t|} + C = \frac{1}{2}\ln\frac{|1+\sin t|^2}{\cos^2 x} + C = \ln|\sec t + \tan t| + C$$
$$= \ln|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}| + C$$

(c) Put$x = \sec t$, then$\sqrt{x^2-1} = \tan t, dx = \sec t\tan t$

$$\int \frac{2}{x^3\sqrt{x^2-1}}dx = 2\int \frac{1}{\sec^3 t\tan t}\sec t\tan tdt = 2\int \cos^2 tdt = \int (\cos 2t + 1)dt$$
$$= \frac{1}{2}\sin 2t + t + C = \frac{\sqrt{x^2-1}}{x^2} + \arctan\sqrt{x^2-1} + C$$

(d) Put $x = 4\sin t$,then $dx = 4\cos tdt$

$$\int x^2\sqrt{16-x^2}dx = \int 16\sin^2 t \cdot 4\cos t \cdot 4\cos tdt = 256\int \sin^2 t\cos^2 tdt$$
$$= 64\int \sin^2 2tdt = 32\int (1-\cos 4t)dt = 32t - 8\sin 4t + C$$
$$= 32\arcsin\frac{x}{4} - 2x\sqrt{16-x^2}\left(1 - \frac{x^2}{8}\right) + C$$

8. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$

(b) $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

(c) $\displaystyle\int \frac{x}{\sqrt{25-x^2}}dx$

(d) $\displaystyle\int x\sin^{-1} xdx$

(e) $\displaystyle\int \tan^3 xdx$

(f) $\displaystyle\int x\sin^2 xdx$

(g) $\int x \sec^2 x dx$ (h) $\int \frac{dx}{1-\cos x}$

Solution:

(a) $\int \frac{dx}{\sqrt{x}(1+x)} = 2\int \frac{1}{1+(\sqrt{x})^2} d(\sqrt{x}) = 2\arctan(\sqrt{x}) + C$

(b) $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x}+1} dx = \int \frac{d(e^x)}{e^{2x}+1} = \arctan(e^x) + C$

(c) Put$x = 5\sin t$, then $\sqrt{25-x^2} = 5\cos t$,$dx = 5\cos t dt$
$\int \frac{x}{\sqrt{25-x^2}} dx = \int \frac{5\sin t}{5\cos t} \cdot 5\cos t dt = 5\int \sin t dt$
$= -5\cos t + C = -\sqrt{25-x^2} + C$

(d) $\int x\sin^{-1} x dx = \int x \arcsin x dx = \int \arcsin x d(\frac{x^2}{2}) = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$
For the integral $\int \frac{x^2}{\sqrt{1-x^2}} dx$, put $x = \sin t$, then $dx = \cos t dt$ so that
$\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{\sin^2 t \cos t}{\cos t} dt = \int \sin^2 t\, dt = \frac{1}{2}\int 1 - \cos 2t dt$
$= \frac{t}{2} - \frac{\sin 2t}{4} + C = \frac{t - \sin t \cos t}{2} + C = \frac{\arcsin x - x\sqrt{1-x^2}}{2} + C$. Thus, we obtain the result $\int x \sin^{-1} x dx = \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right) + C.$

(e) $\int \tan^3 x dx = \int \tan^2 x \cdot \tan x dx = \int (\sec^2 x - 1)\tan x dx$
$= \int \tan x d(\tan x) - \int \frac{\sin x}{\cos x} dx = \frac{1}{2}\tan^2 x + \int \frac{1}{\cos x} d(\cos x)$
$= \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

(f) $\int x\sin^2 x dx = \frac{1}{2}\int x(1-\cos 2x) dx = \frac{1}{4}x^2 - \frac{1}{2}\int x\cos 2x\, dx = \frac{1}{4}x^2 - \frac{1}{4}\int x d(\sin 2x)$
$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x + \frac{1}{4}\int \sin 2x dx = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$

(g) $\int x\sec^2 x dx = \int x d(\tan x) = x\tan x - \int \tan x dx = x\tan x - \int \frac{\sin x}{\cos x}$
$= x\tan x + \int \frac{1}{\cos x} d(\cos x) = x\tan x + \ln|\cos x| + C$

(h) note that $d(-\frac{1}{\tan x}) = \frac{1}{\sin^2 x}$
$\int \frac{dx}{1-\cos x} = \int \frac{1+\cos x}{1-\cos^2 x} dx = \int \frac{1+\cos x}{\sin^2 x} = \int \frac{1}{\sin^2 x} dx + \int \frac{1}{\sin^2 x} d(\sin x)$
$= -\frac{1}{\tan x} - \frac{1}{\sin x} + C$

9. (a) Prove that $\int_0^1 \frac{u^4(1-u)^4}{1+u^2}\,du = \frac{22}{7} - \pi.$

(b) Evaluate $\int_0^1 u^4(1-u)^4\,du$ and hence show that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

Solution:

(a)

$$\begin{aligned}\int_0^1 \frac{u^4(1-u)^4}{1+u^2}\,du &= \int_0^1 \frac{u^8-4u^7+6u^6-4u^5+u^4}{1+u^2}du \\ &= \int_0^1 (u^6-4u^5+5u^4-4\frac{u^4}{1+u^2})du \\ &= \frac{1}{7}-\frac{2}{3}+1-4\int_0^1 \frac{u^4-1}{u^2+1}du - 4\int_0^1 \frac{1}{1+u^2}du \\ &= \frac{1}{7}-\frac{2}{3}+1-4\int_0^1 (u^2-1)du - 4\arctan u|_0^1 \\ &= \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-4\cdot\frac{\pi}{4} \\ &= \frac{22}{7}-\pi\end{aligned}$$

(b)

$$\begin{aligned}\int_0^1 u^4(1-u)^4\,du &= \int_0^1 (u^8-4u^7+6u^6-4u^5+u^4)du \\ &= \frac{1}{9}-\frac{1}{2}+\frac{6}{7}-\frac{2}{3}+\frac{1}{5} = \frac{1}{630},\end{aligned}$$

Since $u^4(1-u)^4 > \frac{u^4(1-u)^4}{1+u^2} > \frac{u^4(1-u)^4}{2} > 0, x \in (0,1)$,then $\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}$, so

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $a \in \mathbb{R}$. Show that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

Hence, evaluate $\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\,dx.$

Solution: Put $t = a - x$, then$x = a - t$,so

$$\int_0^a f(a-x)\,dx = \int_a^0 f(t)d(a-t) = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx$$

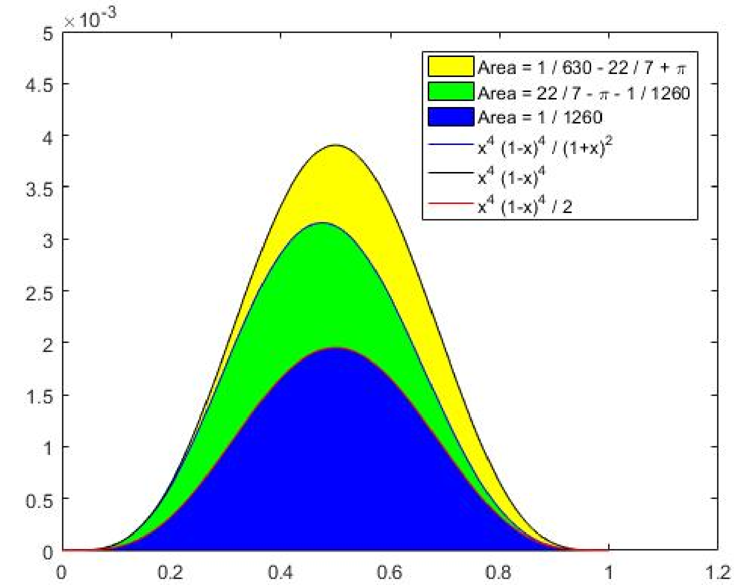

Figure 1: Q9

Since $\frac{\cos^3 x}{\sin x+\cos x}$ is continuous in $[0, \frac{\pi}{2}]$, by above argument, we have

$$\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} \frac{\cos^3(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}\, dx = \int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}\, dx$$

$$\begin{aligned}
2\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\, dx &= \int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}\, dx \\
&= \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}\, dx \\
&= \int_0^{\pi/2} \sin^2 x - \sin x \cos x + \cos^2 x\, dx \\
&= \frac{\pi}{2} - \int_0^{\pi/2} \sin x \cos x dx \\
&= \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi/2} \sin 2x dx \\
&= \frac{\pi}{2} - \frac{1}{2}
\end{aligned}$$

therefore, $\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\, dx = \frac{\pi}{4} - \frac{1}{4}$.

11. (a) Let $f, g : [0, a] \to \mathbb{R}$ be two continuous functions that satisfy

$$f(x) = f(a - x) \qquad \text{and} \qquad g(x) + g(a - x) = M,$$

where M is a real constant. Show that

$$\int_0^a f(x)g(x)\,dx = \frac{M}{2}\int_0^a f(x)\,dx.$$

(b) Hence, evaluate $\int_0^\pi x\cos^2 x\sin^4 x\,dx$.

Solution:

(a) Since f, g are continuous functions on $[0, a]$,and $f(x) = f(a - x)$ and $g(x) + g(a - x) = M$, by Q10

$$\begin{aligned}\int_0^a f(x)g(x)\,dx &= \int_0^a f(a-x)g(a-x)\,dx \\ &= \int_0^a f(x)(M - g(x))\,dx\end{aligned}$$

then,

$$2\int_0^a f(x)g(x)\,dx = M\int_0^a f(x)\,dx$$

hence,

$$\int_0^a f(x)g(x)\,dx = \frac{M}{2}\int_0^a f(x)\,dx.$$

(b) Put $f(x) = \cos^2 x\sin^4 x$, $g(x) = x$, then $f(x) = f(\pi - x)$, $g(x) + g(\pi - x) = \pi$ then

$$\int_0^\pi x\cos^2 x\sin^4 x\,dx = \frac{\pi}{2}\int_0^\pi \cos^2 x\sin^4 x\,dx$$

Since

$$\begin{aligned}\int_0^\pi \cos^2 x\sin^4 x\,dx &= \frac{1}{4}\int_0^\pi \sin^2 x\sin^2 2x\,dx \\ &= \frac{1}{8}\int_0^\pi (1 - \cos 2x)\sin^2 2x\,dx \\ &= \frac{1}{8}\int_0^\pi \sin^2 2x dx - \frac{1}{16}\int_0^\pi \sin^2 2x d(\sin 2x) \\ &= \frac{1}{16}\int_0^\pi (1 - \cos 4x)dx - 0 \\ &= \frac{\pi}{16} - \frac{1}{64}\sin 4x|_0^\pi \\ &= \frac{\pi}{16}\end{aligned}$$

therefore,

$$\int_0^\pi x\cos^2 x\sin^4 x\,dx = \frac{\pi}{2}\int_0^\pi \cos^2 x\sin^4 x\,dx = \frac{\pi^2}{32}.$$

12. (a) Let $f(x)$ and $g(x)$ be two continuous functions on $[a, b]$. For $x \in [a, b]$, let

$$F(x) = \left(\int_a^x [f(t)]^2\, dt\right)\left(\int_a^x [g(t)]^2\, dt\right) - \left(\int_a^x f(t)g(t)\, dt\right)^2.$$

Show that $F(x)$ is increasing on $[a, b]$ and hence deduce that

$$\left(\int_a^b [f(x)]^2\, dx\right)\left(\int_a^b [g(x)]^2\, dx\right) \geq \left(\int_a^b f(x)g(x)\, dx\right)^2.$$

(b) Using the result in (a), or otherwise, show that

$$\ln\left(\frac{p}{q}\right) \leq \frac{p-q}{\sqrt{pq}},$$

where $0 < q \leq p$.

Solution:

(a) Note that $F(x)$ is continuous and differentiable on $[a, b]$,therefore,

$$\begin{aligned} F^{'}(x) &= f^2(x)\int_a^x [g(t)]^2\, dt + g^2(x)\int_a^x [f(t)]^2\, dt - 2f(x)g(x)\int_a^x f(t)g(t)dt. \\ &= \int_a^x f^2(x)[g(t)]^2 - 2f(x)g(x)f(t)g(t) + g^2(x)[f(t)]^2\, dt \\ &= \int_a^x [f(x)g(t) - f(t)g(x)]^2\, dt \geq 0 \end{aligned}$$

Hence, $F(x)$ is increasing on $[a, b]$,then $F(b) \geq F(a) = 0$, that is.

$$\left(\int_a^b [f(x)]^2\, dx\right)\left(\int_a^b [g(x)]^2\, dx\right) \geq \left(\int_a^b f(x)g(x)\, dx\right)^2.$$

(b) Put $f = 1,\ g = \frac{1}{x}$, by(a),

$$\ln(\frac{p}{q}) = \int_q^p \frac{1}{x}dx \leq \left(\int_q^p 1dx\right)^{\frac{1}{2}}\left(\int_q^p \frac{1}{x^2}dx\right)^{\frac{1}{2}} = (p-q)^{\frac{1}{2}}(\frac{1}{q} - \frac{1}{p})^{\frac{1}{2}} = \frac{p-q}{\sqrt{pq}}.$$

13. Let $f(x)$ be a continuous function on $[0, a]$.

(a) Prove that $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

(b) Prove that $1 + \tan\left(\frac{\pi}{4} - x\right) = \dfrac{2}{1 + \tan x}$.

(c) Prove that $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1 + \tan x) = \frac{\pi \ln 2}{8}$.

Solution:

(a) Put $t = a - x$, then$x = a - t$, so

$$\int_0^a f(a-x)\,dx = \int_a^0 f(t)d(a-t) = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx$$

(b) Observe that

$$\tan(\frac{\pi}{4} - x) = \frac{1 - \tan x}{1 + \tan x},$$

so

$$1 + \tan(\frac{\pi}{4} - x) = \frac{1 + \tan x + 1 - \tan x}{1 + \tan x} = \frac{2}{1 + \tan x}.$$

(c) By the previous two arguments, observe that

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln 2 - \ln\left(1 + \tan(\frac{\pi}{4} - x)\right)\mathrm{d}x \\ &= \frac{\pi \ln 2}{4} - \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan(\frac{\pi}{4} - x)\right)\mathrm{d}x,\end{aligned}$$

and

$$\int_0^{\frac{\pi}{4}} \ln\left(1 + \tan(\frac{\pi}{4} - x)\right)\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x.$$

Hence,

$$\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x = \frac{\pi \ln 2}{8}.$$