THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2020-2021 Term 1
Homework Assignment 4
Due Date: 7 December 2020 (Monday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ____________________ Date ____________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.
- Late assignments will receive a grade of 0.
- Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty.
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Evaluate the following integrals:

(a) $\int \frac{x}{\sqrt{1+x^2}} dx$

(b) $\int \frac{x^5}{(1+x^3)^3} dx$

(c) $\int \frac{1}{x^2} \sin \frac{1}{x} \, dx$

(d) $\int \frac{x-2}{\sqrt{x^2-4x+3}} dx$

(e) $\int x^2 \sqrt{x^3+2} \, dx$

(f) $\int \sin^5 x \cos x dx$

(g) $\int \sin 3x \sin 5x dx$

(h) $\int \cos x \cos 7x \, dx$

(i) $\int \sin^2 2x \sin 5x \, dx$

(j) $\int \cos^2 2x \sin^3 2x \, dx$

2. Evaluate the following indefinite integrals.

(a) $\int x^2 \ln x dx$

(b) $\int x \sec^2 x dx$

(c) $\int e^{-x} \sin 3x \, dx$

(d) $\int \sin(\ln x) dx$

3. Find $F'(x)$ for the following functions.

(a) $F(x) = \int_{\pi}^{x} \frac{\cos y}{y} \, dy$

(b) $F(x) = \int_{0}^{x^3} e^{u^2} \, du$

(c) $F(x) = \int_{x}^{2x} (\ln t)^2 \, dt$

(d) $F(x) = \int_{x^2}^{x^3} e^{\cos u} \, du$

(e) $F(x) = \int_{1}^{x} \frac{e^x + e^t}{t} dt$

(f) $F(x) = \int_{1}^{x} \frac{e^{xt}}{t} dt$

4. Evaluate the following definite integrals.

(a) $\int_0^1 x^3 \sqrt{1+3x^2} dx$

(b) $\int_0^{\pi} x \sin 2x \, dx$

(c) $\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx$

(d) $\int_0^5 |x^2 - 4x + 3| \, dx$

5. Prove the following reduction formulas.

(a) $I_n = \int \sin^n x dx;\ I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2},\ n \geq 2$

(b) $I_n = \int (\ln x)^n dx;\ I_n = x(\ln x)^n - nI_{n-1},\ n \geq 1.$

(c) $I_n = \int x^n \cos x dx;\ I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2},\ n \geq 2$

(d) $I_n = \int \frac{dx}{(x^2-a^2)^n};\ I_n = -\frac{x}{2a^2(n-1)(x^2-a^2)^{n-1}} - \frac{2n-3}{2a^2(n-1)} I_{n-1},\ n \geq 1$

6. Evaluate the following integrals of rational functions.

(a) $\int \frac{x^2 dx}{1-x^2}$

(b) $\int \frac{x^3}{x+3}\, dx$

(c) $\int \frac{4x+1}{x^2-6x+13} dx$

(d) $\int \frac{(1+x)^2}{1+x^2}\, dx$

(e) $\int \frac{2x^3-x^2+3}{x^2-2x-3}\, dx$

(f) $\int \frac{6x+11}{(x+1)^2}\, dx$

(g) $\int \frac{x^2+1}{(x+1)^2(x-1)},\, dx$

(h) $\int \frac{x^4}{x^2-4}\, dx$

(i) $\int \frac{dx}{(x+1)(x^2+1)}$

(j) $\int \frac{dx}{x(x^2+1)^2}$

7. Evaluate the following integrals by trigonometric substitution.

(a) $\int \frac{x^2 dx}{(1-x^2)^{\frac{3}{2}}}$

(b) $\int \frac{dx}{\sqrt{4+x^2}}$

(c) $\int \frac{2}{x^3\sqrt{x^2-1}}\, dx$

(d) $\int x^2\sqrt{16-x^2} dx$

8. Evaluate the following integrals.

(a) $\int \frac{dx}{\sqrt{x}(1+x)}$

(b) $\int \frac{dx}{e^x+e^{-x}}$

(c) $\int \frac{x}{\sqrt{25-x^2}} dx$

(d) $\int x \sin^{-1} x dx$

(e) $\int \tan^3 x dx$

(f) $\int x \sin^2 x dx$

(g) $\int x \sec^2 x dx$

(h) $\int \frac{dx}{1-\cos x}$

9. (a) Prove that $\displaystyle\int_0^1 \frac{u^4(1-u)^4}{1+u^2}\, du = \frac{22}{7} - \pi.$

(b) Evaluate $\displaystyle\int_0^1 u^4(1-u)^4\, du$ and hence show that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $a \in \mathbb{R}$. Show that

$$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$$

Hence, evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\, dx.$

11. (a) Let $f, g : [0, a] \to \mathbb{R}$ be two continuous functions that satisfy

$$f(x) = f(a-x) \qquad \text{and} \qquad g(x) + g(a-x) = M,$$

where M is a real constant. Show that

$$\int_0^a f(x)g(x)\,dx = \frac{M}{2}\int_0^a f(x)\,dx.$$

(b) Hence, evaluate $\displaystyle\int_0^{\pi} x\cos^2 x\sin^4 x\,dx$.

12. (a) Let $f(x)$ and $g(x)$ be two continuous functions on $[a,b]$. For $x \in [a,b]$, let

$$F(x) = \left(\int_a^x [f(t)]^2\,dt\right)\left(\int_a^x [g(t)]^2\,dt\right) - \left(\int_a^x f(t)g(t)\,dt\right)^2.$$

Show that $F(x)$ is increasing on $[a,b]$ and hence deduce that

$$\left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right) \ge \left(\int_a^b f(x)g(x)\,dx\right)^2.$$

(b) Using the result in (a), or otherwise, show that

$$\ln\left(\frac{p}{q}\right) \le \frac{p-q}{\sqrt{pq}},$$

where $0 < q \le p$.

13. Let $f(x)$ be a continuous function on $[0,a]$.

(a) Prove that $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

(b) Prove that $1 + \tan\left(\frac{\pi}{4} - x\right) = \dfrac{2}{1+\tan x}$.

(c) Prove that $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan x) = \frac{\pi\ln 2}{8}$.